Privacy in Bargaining and Audience Intervention*

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February 20, 2020

Abstract

A seller makes offers every instant, without commitment, to a privately informed buyer. An audience observes the negotiation and can choose to intervene in the seller's favor at a cost. The audience observes all offers (public bargaining) or only the bargaining delay but not the offers (private bargaining). If the audience prefers intervening in the seller's favor when the buyer's type is lower, then typically private bargaining benefits the seller, raises prices, and increases delay. If the audience prefers intervening in the seller's favor when the type is higher, the effects are reversed.

*Uncountably many thanks to my advisors, Andy Skrzypacz, Matt Jackson, and Kyle Bagwell, for their guidance and encouragement. I would also like to thank Omer Ali, Alex Bloedel, Gabriel Carroll, Piotr Dworczak, Yingni Guo, Shota Ichihashi, Giorgio Martini, Guido Martirena, Dmitry Orlov, Larry Samuelson, Bruno Strulovici, Felipe Varas, Pavel Zryumov, and audiences at various seminars and conferences for insightful discussions. I gratefully acknowledge support from the Kapnick Foundation Fellowship, granted through the Stanford Institute for Economic Policy Research. Matteo Camboni provided excellent research assistance.

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1 Introduction

This paper studies the effects of privacy on bargaining. How do agents negotiate differently when they know their offers are being observed? I study the *intervention motive* for privacy: as they bargain, agents worry about strategic audiences who observe (something about) the negotiation and whose actions change the nature of the negotation and the parties' relative bargaining power.[1]

- Private Politics: When a labor union bargains with a firm over a wage contract, both parties understand that there are regulators observing the negotiation who can choose to intervene depending on how the bargaining evolves—regulators can impose an arbitration agreement, they can force injunctions that prevent lockouts, or they can require the union to continue working before an agreement is reached. For example, the early 2000's strike by ILWU, the longshoremens union in the US, essentially ended after the intervention of President Bush. At the time, both parties were clearly aware of the possibility of endogenous interruption, and in fact press reports suggest that the shippers's bargaining strategies were tailored explicitly to induce the president's intervention. According to a New York Times article on the dispute, "[t]he union said the lockout was a management ploy intended to have the president intervene." The AFL-CIO secretary-treasurer said, "[w]e're absolutely furious . . . The P.M.A. [ports management] locked the workers out, contrived a phony crisis and then gets rescued by the administration. They're getting their way and have the weight of the government behind them."[2]
- M&A: Many mergers-and-acquisitions deals start as bilateral negotiations between a company and an acquirer, but they endogenously become auctions when another acquirer chooses to trigger a bidding war after its interest is piqued. To my knowledge, this process of negotiations endogenously becoming auctions has not been tabulated explicitly in the corporate finance literature, but there is some indicative evidence that it is common. Schwert (1996) shows that, in the 1975-1991 period, 30% of tender offers for US-listed companies in his sample eventually drew multiple bidders.
- A variation on the former example is that large corporate negotiations attract media scrutiny, which can eventually generate publicity that changes the payoffs of the different players. Media attention, for example, will strengthen the hand of activist investors who need to corral the votes of many dispersed shareholders in order to impose their views on management.
- International relations: Disputes between countries take place under the scrutiny of domestic audiences (see, e.g., the rich political science literature going back to Fearon (1994); see also Schultz (1998)). These audiences respond to the parry of offers and counter-offers and can then impose political costs on the governments conducting the negotiations.[3] At an extreme, some bargaining posture taken abroad could bring about the collapse of a government, so both countries in a negotiation will adapt their bargaining to the possibility of intervention by their domestic publics. Fearon (1994) reports the following remark made by the French minister of foreign affers to the British ambassador in the buildup to the Seven Years' War (1757-1763):

[1] Of course this is not the only motive for privacy. Parties could also have reputational concerns, i.e., they worry about audiences that matter outside of the current negotiation.

[2] See `www.nytimes.com/2002/10/09/us/president-invokes-taft-hartley-act-to-open-29-ports.html`.

[3] The political science literature typically takes these audience costs as exogenous and explores their role in allowing countries to engage in costly signalling of true intentions. The focus here is rather on the audience's *endogenous* decision to impose costs and on how bargaining strategies respond to the possibility of that decision under different information structures.

> [The minister] complaine'd very much of the licentiousness of our Publick papers in exaggerating things beyond measure which only served to irritate and stir up animosity amongst the lower sort of People in both Nations without a just cause.

In some of these examples, such as the president in the ILWU strike, the audience *adjudicates* in favor of one of the parties at a moment of crisis. In others, such as the media and M&A example, the audience *interrupts* the negotiation. What all examples have in common is that the negotiating parties would incorporate concerns about endogenous intervention into their bargaining behavior.

Abstracting somewhat from institutional detail, I study these issues in a model where a seller and a privately informed buyer bargain in the shadow of an audience who may stochastically choose to intervene. Time is continuous, on an infinite horizon, and the buyer and seller both discount time at the same rate. The seller makes offers to the buyer, who has private information about his value for the good. The seller can adjust her offers every instant, without commitment.[4] The audience receives a single decision opportunity at a Poisson rate ℓ_0; when such an opportunity arises, the audience can decide between a costly option, $d = Y$, and a free one, $d = N$. By convention, I assume that the costly option is more favorable to the seller. The audience's net payoffs from this choice depend on the buyer's true type, so that as the parties negotiate, they realize that the informational content of their offers affects the audience's decision.

By re-interpreting the players and the arrival of decision opportunities, the model can accomodate both adjudication and interruption scenarios:

Adjudication: The audience is long-lived but can only intervene upon the arrival of a crisis that arrives at a Poisson rate. At that time, the audience must *adjudicate*, i.e., rule in favor of either the seller or the buyer, where ruling for the seller incurs an additional random cost. The game then ends, giving all parties their intervention payoffs.

Interruption: The audience consists of a stream of short-lived audiences who arrive at a Poisson rate. Each audience, when it arrives, chooses between interrupting the negotiation at a cost, or leaving. If a time-t audience leaves, the game continues as before; if instead it iterrupts, the game ends and players receive their intervention payoffs.

Having the game end after intervention need not be interpreted literally; intervention can also represent the start of a different continuation game, and the intervention payoffs will be sufficiently flexible to include the reduced form of many different continuation games. The key restriction is that the audience cannot optimally choose the timing of intervention—it must take the opportunity when it arises or let it go forever. While the proofs lean heavily on this feature, the rankings I provide are strict, so I do not believe this to be a knife-edge case. The assumption of a single, non-deferrable intervention opportunity would mean, for the regulatory example, that intervention spends down the regulator's precious political capital, and as such is only done in times of exception where the regulator's meddling would not be seen as controversial. (Alternatively, the regulator could be inattentive, and it is only the crisis that highlights the need for intervention.) Domestic political audiences, in the international dispute example, are plausibly more reactive and operate on a shorter horizon than the negotiating governments, so the assumption may not be unduly restrictive there. In the M&A bidding-war example, the restriction means that the model is better suited to the case where entering buyers are either opportunistic buyers not

[4] Throughout, I refer to the seller as "she," and to the incumbent buyer as "he" whenever this would not cause confusion.

usually in that market, or part of a competitive fringe and therefore on a different timeline than the incumbent buyer.

I study how outcomes differ when bargaining happens in public (the audience sees the entire history of rejected offers, "public game" for short) versus when it happens in private (the audience sees how long bargaining has been going on for, but not the specific offers that have been made and rejected, called the "private game" for short). In the private game, the buyer and the seller can deviate in ways that are unobservable to the audience, leading to different equilibrium behavior than the public game. Because of the discounting, higher types of the buyer dislike delay relatively more, and the game satisfies the usual "skimming property": higher types accept any offer that lower types would have accepted, and after any history, the seller's beliefs are truncations of her prior. I study equilibria that are Markovian in that truncation cutoff.

The main force in the model is that *disagreement affects the discounted probability of costly intervention.* The audience, knowing that high types of the buyer dislike delay relatively more, interprets every offer that the buyer rejects as further evidence that the buyer's value is low (since a high enough type would have accepted). If the audience's payoffs net payoff from costly intervention are increasing in the buyer's type, disagreement discourages (costly) intervention (DDI for short); if on the contrary the audience's intervention payoffs are *decreasing* in the buyer's type, disagreement encourages (costly) intervention (DEI for short).

The main result is that the seller's preferences over publicity depend on whether DDI and DEI holds, and little else. Under DDI, the seller prefers public to private bargaining. Under DEI, the comparison is slightly more subtle, but the seller still broadly prefers privacy: either she prefers private bargaining to public bargaining outright, or she prefers *nearly private* bargaining, where all offers but the first are private, to purely public bargaining. Most of the other fine details of the model—whether the audience adjudicates or interrupts, the precise nature of the post-intervention game or the distribution of buyer's type, etc—will not affect that preference.

Under some additional conditions, I show that the buyer and the seller have a conflict of interest at the ex ante stage. Suppose that the distribution of buyer values is log-concave and the audience's net payoffs satisfy a "diminishing marginal response to information" condition: changes in the cutoff affect the speed of intervention more when the cutoff is high, early in the game, than when the cutoff is low. If intervention is *bilaterally neutral*–intervention neither creates nor destroys the joint value enjoyed by the buyer-seller pair—then under DEI, bargaining in public grows the buyer's ex ante utility *and* the sum of the buyer and the seller's ex ante utilities. Under DDI, the opposite is true. The seller's preferences for privacy in DEI and publicity in DDI thus come at the expense of both the buyer and the pair's bilateral surplus.

Notice that, when she bargains in public, the seller acquires a lever that is unavailable to her in the private game: she can use her offers to modulate the (discounted) probability of a costly intervention. By making lower offers publicly, she can screen the buyer faster and change the audience's beliefs faster (vice-versa for higher offers). This lever would appear, at first blush, to be clearly helpful to the seller. The main result shows this logic is incorrect, and in fact, having this lever available may often *harm* the seller.

The gist of the argument is that the seller's lack of commitment pushes the seller's utility down to what she could obtain by making unreasonable offers forever—a kind of outside option—and the outside option in the public game is better than the private one under DDI and worse under DEI.

Consider the DDI case (the argument for DEI is symmetric). Indeed, if the seller deviates and makes unreasonable offers forever, this freezes the audience's beliefs in the public game (since the audience sees the deviation), but has no effect on those beliefs in the private game. Therefore, this "stalling" deviation leads to a constant rate of costly intervention under public offers, but a dropping rate of costly intervention under private offers. Since the deviation will eventually give the seller the intervention payoffs and no more, then the seller prefers a steady intervention rate to a dropping one, since it delivers those payoffs sooner.

If disagreement encourages costly intervention (recall this is the intervention that favors the seller), it is as though the seller gave the buyer the following speech while bargaining in private:

> Listen, I know I'm supposed to make you a reasonable offer, but you better agree to this higher price right now. If you don't, you're only making things worse for yourself. The audience does not see what I offered you, so if you reject, they will conclude that you rejected a reasonable offer. They will be more willing to intervene in my favor in the future, strengthening my hand even more.

When disagreement *discourages* costly (and seller-favorable) intervention, the tables are turned, and it is as though the *buyer* can make the following speech while bargaining in private:

> Listen, I know you're supposed to make me a reasonable offer, but you better offer me something better. If you don't, you're only making things worse for yourself. The audience does not see what you offer, so if I reject, they will conclude that I rejected something reasonable. They will be less willing to intervene in your favor in the future, weakening your hand even more.

The above intuition for the seller's preferences uses little of the structure of equilibria. In subsequent results I decompose the payoff effects into their price and delay components, and I show that the "price" channel dominates: the seller prefers the regime with higher prices. To boot, under DEI, transaction prices are uniformly higher in the private game, and under DDI, they are uniformly *lower* in the private game. Intuitively, under DEI, the seller makes very low offers in the hope that the arrival of an audience will strengthen her hand, while under DDI, she makes very high offers in order to "protect" the strong hand that she has. If the buyer's values are log-concave and the audience's payoffs satisfy the "diminishing marginal response to information" condition above, then delay and price responds in opposite ways to public bargaining: trade is uniformly faster in public under DEI and uniformly *slower* in public under DDI.

The paper is organized as follows. Section 2 discusses related literature. Section 3 introduces the model and continuous-time equilibrium concept and related formalism. Section 4 formally states the main results on the seller's induced preferences over privacy, which are then developed in the rest of the paper. Section 5 describes the key steps in the proof of the main result. Section 6 decomposes the payoff comparisons into price and delay changes. Section 7 discusses some implications for the equilibrium effect of disclosure policies. It also discusses the contrast between this model and a static one-shot version of the model, highlighting the new insights that come from modeling the lack of commitment. Finally, the section discusses extensions that are covered in the appendices–especially to the gap case, which has more complicated dynamics. Omitted proofs are in the appendices.

2 Related Literature

This work contributes to a recent literature on the effects of price transparency on market performance (Hörner and Vieille, 2009; Kaya and Liu, 2015; Fuchs et al., 2016). This family of models focuses on discrete-time search markets, where a long-run informed player faces offers by a sequence of short-run uninformed players. My paper shares the key concerns of this literature. However, by looking at bargaining between long-lived agents (rather than a search market) and by introducing strategic audiences, I uncover a very different mechanism through which price transparency can shape outcomes: privacy changes the severity of the offering player's commitment problem. I discuss the contrast between my results and those in this literature in Remark 1. In particular, the fact that privacy can have starkly different effects depending on the valuation structure on the buyer's side is new. [5]

In addition, this paper contributes to prior work on the interaction between endogenous entry and private information in markets. An earlier literature on static mechanism design considered how the need to induce the endogenous entry of agents could completely change the auction format a seller should use (McAfee and McMillan, 1987; Levin and Smith, 1994; Ye, 2007). Recent papers in mechanism design consider the new incentive issues that arise when long-lived agents can choose the time at which they arrive to or depart from the market (Garrett, 2017; Mierendorff, 2016; Gershkov et al., 2017). The seller has full commitment power in these papers, so the main force I identify (the interaction between market transparency and the seller's lack of commitment) is absent.[6]

Finally, this paper contributes to a recent literature on bargaining in changing environments (Inderst, 2008; Fuchs and Skrzypacz, 2010; Ortner, 2017; Daley and Green, 2020). In these works, the environment evolves exogenously, due to arrivals of news (Daley and Green, 2020), arrivals of other players (Inderst, 2008; Fuchs and Skrzypacz, 2010), or because the seller's cost changes over time (Ortner, 2017). In common with these papers, I look at Stationary/Markovian equilibria of a Coasian Bargaining setup (Bulow, 1982; Stokey, 1981) in the "frequent offers" limit (Fudenberg et al., 1985; Gul et al., 1986).[7] The "stage game" I use borrows from Fuchs and Skrzypacz (2010), who study the frequent-offers limit of a discrete-time bargaining game where some event that ends the negotiation can arrive at a constant, exogenous rate. My approach to formulating the bargaining interaction directly in continuous time, writing the uninformed agent's problem in "quantity space," follows Ortner (2017) and Daley and Green (2020). Notably, my model has a continuum of types and no driving Brownian motion, so one contribution is providing the right restriction on strategy spaces that allows for tractable analysis of necessary conditions. In addition, the private game here is non-stationary.

[5] To my knowledge, the earliest model on the impact of privacy in bargaining between long-lived agents is Perry and Samuelson (1994). In their two-period model, the "audience" is a constituency on behalf of whom one of the agents is bargaining. When the bargaining happens publicly ("open-door"), the constituency can force an end to the negotiation after observing the failure of the first round of offers, whereas with private bargaining ("closed-door"), the constituency can only accept or reject final offers. The motive for privacy is therefore different from, and complementary to, the one highlighted by my model.

[6] Few works study private information and endogenous entry in dynamic settings without commitment. A notable exception is Zryumov (2015), in which there are good and bad entrepreneurs, and bad entrepreneurs can strategically choose when to enter a competitive market for funding.

[7] See Ausubel et al. (2002) for a comprehensive review of the preceding Coase Conjecture (Coase, 1972) literature.

3 Model and Equilibrium Notions

There is a long-lived seller S and a long-lived buyer B bargaining over an indivisible asset owned by S. B's willingness to pay for the asset is a privately known type $v_B \sim F_B[\underline{v}, 1]$, $\underline{v} \geq 0$. They bargain in the shadow of an audience A that observe (aspects of) the negotiations and can choose to intervene. Time is continuous, with an infinite horizon, and the buyer and the seller discount the future at rate r. Hence, if they agree on a price p at time t *before* the audience intervenes, their payoffs are $e^{-rt}p$ for the seller and $e^{-rt}(v_B - p)$ for the buyer.

The audience may only intervene when an opportunity arises, which happens at a constant Poisson rate ℓ_0. Faced with an opportunity, the audience must then make a decision $d \in \{Y, N\}$, where by convention $d = Y$ is the decision that is "more costly" to the audience. A decision d gives player $i \in \{S, B, A\}$ present-value expected payoffs of $\pi_i^d(v_B)$ as a function of the buyer's type. When it chooses $d = Y$, the audience also suffers a private cost $c \sim G$, which is drawn independently of v_B and the history of the game. $\pi_i(v_B) := \pi_i^Y(v_B) - \pi_i^N(v_B)$ (without a superscript) denotes the net gain to i from intervention in the seller's favor. Below, upper case π's denote the payoffs from intervention *conditional on knowing* $v_B \leq k$: $\Pi_i^d(k) := F_B(k)^{-1} \int_0^k \pi_i^d(v)dF_B(v)$ for $i \in \{S, A\}$, with $\Pi_i(k) := \Pi_i^Y(k) - \Pi_i^N(k)$ (no superscript) once again denoting the net gain from costly intervention.

The model nests two classes of models for the audience decision, *adjudication* and *interruption*:

Adjudication: There is a single audience observing the bargaining game. At rate ℓ_0, a crisis arrives that requires the audience to adjudicate in favor of either the seller ($d = Y$) or the buyer ($d = N$). (The audience cannot intervene before the moment of adjudication). The game ends after the audience's decision.

Interruption: A stream of short-run audiences arrives to the market at rate ℓ_0. Each audience, when it arrives, draws c and chooses whether to interrupt ($d = Y$) or not ($d = N$). The cost c for $d = Y$ is drawn independently across audiences. $\pi_i^N(v_B) = 0$ for all i, v_B. If the time t audience chooses $d = N$, it leaves, and the game continues as before. If the audience chooses $d = Y$ ("interrupting"), the game ends, and the intervention payoffs of $\pi_i^Y(v_B)$, $i \in \{S, B, A\}$ are realized. Figure 1 depicts the interruption "stage game" being played in each $[t, t + dt)$.

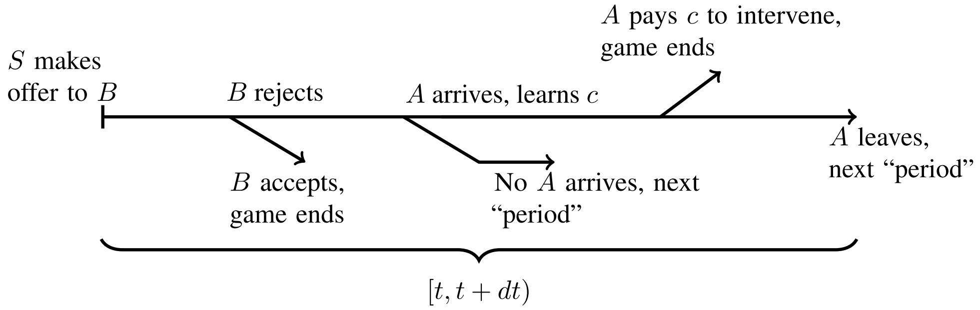

Figure 1: Heuristic "Stage Game", Interruption Case

Given a path of prices $\mathbf{P} = \{P_t\}_{t \geq 0}$, if the audience intervenes at time σ and makes a decision d_σ, a buyer with type v_B who plans to accept an offer at τ would receive ex post payoffs of

$$e^{-r\tau}\mathbf{1}_{\tau \geq \sigma}(v_B - P_\tau) + e^{-r\sigma}\mathbf{1}_{\tau > \sigma}\left(\mathbf{1}_{d_\sigma = Y}\pi_B^Y(v_B) + \mathbf{1}_{d_\sigma = N}\pi_B^N(v_B)\right)$$

while the seller receives (given the type v_B of the buyer)

$$e^{-r\tau}\mathbf{1}_{\tau\geq\sigma}P_\tau + e^{-r\sigma}\mathbf{1}_{\tau>\sigma}\left(\mathbf{1}_{d_\sigma=Y}\pi_S^Y(v_B) + \mathbf{1}_{d_\sigma=N}\pi_S^N(v_B)\right)$$

I impose the following conditions.

Assumption 1 (Smooth Primitives)**.** F_B and G have full-support, strictly positive continuous densities $f_B \in C^2[0,1]$ and $g \in C^3[0,\bar{c}]$.[8] $\pi_i^d, i \in \{S,B,A\}$, $d \in \{Y,N\}$ are C^2 functions.

Assumption 2.

1. $\pi_B'(k), (\pi_B^N)'(k) \leq 1$.
2. $\pi_S'(k), (\pi_S^N)'(k) \geq 0$, with strictly inequality for $k < 1$.
3. $\pi_A'(k) \neq 0$, with constant sign.

Item 1 ensures the typical "skimming" property, whereby high types dislike delay relatively more and accept sooner (Lemma 2 in the appendix). Posterior beliefs about v_B that are then right-truncations of the prior, and the buyer's acceptance decisions can be summarized with a path of truncations ("cutoffs") $\{K_t\}_{t\geq 0}$. Item 2 states that the seller prefers high types of the buyer, and more so when the decision is costly. In this sense, *the costly decision is favorable to the seller*. Item 3 says that the audience always responds to new information about the buyer in a monotone way: when v_B is rises, it always either becomes more willing to choose $d = Y$, or always becomes less willing.

Buyer and Seller Problems With some notation for the present value of a dollar that pays at time t *if the game has not ended yet*, we can write the players' expected payoffs from any adjudication or intervention game in a unified way. Let λ_t be the time-t instantaneous rate at which the game ends with a costly decision by the audience, i.e., $\lambda_t dt \approx \mathbb{P}(\text{game ends during } [t, t+dt), d = Y)$, and denote with boldface $\boldsymbol{\lambda} = \{\lambda_t\}_{t\geq 0}$ a path of such rates. Then, as a (possibly trivial) function of $\boldsymbol{\lambda}$, the present value of a dollar that pays at time t if the game has not ended is

$$\delta(t;\boldsymbol{\lambda}) = e^{-rt}\mathbb{P}(\text{game not over by } t) = \begin{cases} e^{-rt}e^{-\int_0^t \lambda_\nu d\nu}, \text{for interruption games} \\ e^{-rt}e^{-\ell_0 t}, \text{for ajdudication games.} \end{cases}.$$

The buyer's expected payoffs from accepting at τ, given the other player's actions, are

$$\mathcal{U}_B^{v_B}(\tau, \mathbf{P}, \boldsymbol{\lambda}) = \delta(\tau;\boldsymbol{\lambda})(v_B - P_\tau) + \int_0^\tau \delta(t;\boldsymbol{\lambda})[\lambda_t\pi_B(v_B) + \ell_0\pi_B^N(v_B)]dt \tag{1}$$

The expected payoffs to the seller from a path of prices $\mathbf{P}$, given rates of costly intervention $\boldsymbol{\lambda}$, and buyer acceptance decisions summarized by a cutoff path $\mathbf{Q}$, as

$$\mathcal{U}_S(\mathbf{Q}, \mathbf{P}, \boldsymbol{\lambda}) = \int_0^\infty \delta(t;\boldsymbol{\lambda})P_t d(1-F_B(Q_t)) + \int_0^\infty \delta(t;\boldsymbol{\lambda})F_B(Q_t)\left[\lambda_t\Pi_S(Q_t) + \ell_0\Pi_S^N(Q_t)\right]dt$$

Key Preliminaries: Effects of Information Let $\Lambda(k) := \ell_0 G(\Pi_A(k))$; by standard Poisson sampling results, this is the rate at which an arrival of an opportunity results in a costly intervention, if the state is permanently held at k.

Definition 1 (Discount function)**.** For a path of costly intervention rates $\boldsymbol{\lambda} = \{\lambda_t\}_{t\geq 0}$, let

[8] That is, they are C^2 and C^3 in the interior of their respective domain, their right-hand derivatives exist and are continuous at left corners, and vice-versa for right corners.

$$D^X(t|\boldsymbol{\lambda}) := \frac{1}{\delta(t|\boldsymbol{\lambda})}\int_t^\infty \lambda_s \delta(s|\boldsymbol{\lambda})ds \tag{2}$$

denote the t-present value of a claim that pays $1 dollar at the moment of a costly intervention, given that the game has not ended by t. Then the **discount function at cutoff** k is the expected discount given a *constant* rate of costly intervention $\lambda_{t'} = \Lambda(k) \forall t' \geq t$:

$$D(k) := D^X(t|\{\Lambda(k)\}_{t\geq 0}) \tag{3}$$

(note that the right hand side is always constant in t).

Definition 2 (DEI/DDI)**.** Let D be the discount function in (3). Say "disagreement encourages (costly) intervention," abbreviated DEI" if $D' < 0$, and "disagreement discourages (costly) intervention" abbreviated DDI, if $D' > 0$.

DEI (DDI) is solely a property of the audience's payoffs: $D' < 0$ whenever $\pi'_A < 0$, i.e., learning that the buyer's type is lower makes the audience more likely to intervene in the seller's favor (vice versa for $D' > 0$).

Next, I impose a condition that rules out some trivial dynamics:

Assumption 3.

1. Let

$$\begin{aligned}
\hat{P}(k) &:= D(k)\pi_S(k) - D'(k)\frac{F_B(k)}{f_B(k)}\Pi_S(k) + \frac{\ell_0}{\ell_0 + r}\pi_S^N(k) &(4)\\
\underline{P}(k) &:= \max\left\{\hat{P}(k), \max\{D(\underline{v}), D(1)\}\pi_S(k) + \frac{\ell_0}{\ell_0+r}\pi_S^N(k)\right\}\\
\overline{P}(k) &:= k - \max\{D(\underline{v}), D(1)\}\pi_B(k) - \frac{\ell_0}{\ell_0+r}\pi_B^N(k)
\end{aligned}$$

   Then $\underline{P}(k) \leq \overline{P}(k)$, with strict inequality for $k > \underline{v}$.
2. $D(1)\Pi_S(1) + \frac{\ell_0}{\ell_0+r}\Pi_S^N(1) > \underline{v} - D(1)\pi_B(\underline{v}) - \frac{\ell_0}{\ell_0+r}\pi_B^N(\underline{v})$.

The expressions in Assumption 3.1 may seem obscure at first blush, but in fact they are only a "gains from trade" assumption in disguise. To see this, assume for a moment that intervention is exogenous and always in favor of the seller. Then $D(k) = \frac{\ell_0}{\ell_0+r}$ and the condition simplifies to

$$\frac{\ell_0}{\ell_0+r}\pi_S(k) \leq k - \frac{\ell_0}{\ell_0+r}\pi_B(k) \tag{GAINS}$$

The left hand side is the *lowest price* at which the seller is induced to trade when she knows that the buyer's type is k; it equals her "reservation payoff" from stalling forever. The buyer's reservation payoff in this simplified game is $\frac{\ell_0}{\ell_0+r}\pi_B(k)$, so the right hand side is the *highest price* at which the buyer can be induced to trade. Hence (GAINS) says that the lowest price at which the seller is willing to trade is lower than the highest price at which the buyer is willing to trade, i.e., there are gains from trade.

Assumption 3.2 rules out instant trade. It says that, at the start of the game, the seller would rather wait for intervention than make a kind of "pooling" offer that all types would accept: if the audience believes the seller is stalling, then the right hand side of Condition 2 is $\underline{v}$'s reservation price.

Assumptions 2 and 3 are the key economic restrictions. To streamline the analysis, I impose an additional restriction that greatly simplifies the equilibrium description.

Definition 3. The game has a *gap* if $\underline{P}(\underline{v}) < \overline{P}(\underline{v})$, and *no gap* if $\underline{P}(\underline{v}) = \overline{P}(\underline{v})$.

Assumption 4.

1. There exists $k^\dagger \in (0,1]$ such that $\hat{P}(k)$ defined in (4) is strictly increasing for $k \leq k^\dagger$, and $\hat{P}(k) \leq \hat{P}(k^\dagger) \forall k$.
2. There is no gap.

Assumption 4.1 will imply uniqueness of public equilibria. $\hat{P}$ will satisfy this, for instance, if it is single-peaked.[9] Appendix B and Online Appendix A.4 show that the bulk of the analysis goes through when the assumption is removed.

For concreteness, I often refer to the following interruption and adjudication examples:

Example 1 (Interruption: competitors & bidding wars)**.** The audience consists of a competitive fringe of other buyers, called "entrants" for short. If an entrant intervenes, a bidding war is immediately triggered between the two buyers. The costly intervention cost c is a *due diligence* cost: by paying $c \sim G[0, \bar{c}]$, the entrant privately learns a signal v_A relevant to its value. The entrant's value is $\xi(v_A, v_B)$, with $\frac{\partial \xi}{\partial v_B} \in (0,1)$ and $v_A \sim F_A(\cdot|v_B)$, so there may be a common value component as well as correlated signals. The c's and v_A's are independent across entrants (conditional on v_B) and independent of the Poisson arrival process. The payoffs $\pi_i(v_B)$ are the reduced form payoffs from the bidding war. For instance, if the bidding war is a full-information ascending auction with no reserve, and one focuses on the natural ex post equilibrium (Milgrom and Weber, 1982)[10] then $\pi_B(v_B) = \int (v_B - \xi(v_B, v_A))\mathbf{1}_{\omega(v_A) \leq v_B} dF_A(v_A|v_B), \pi_S(v_B) = \int \min\{v_B, \omega(v_A)\} dF_A(v_A|v_B),$, and $\pi_A(v_B) = \int (\xi(v_A, v_B) - v_B)\mathbf{1}_{\omega(v_A) > v_B} dF_A(v_A|v_B)$, where $\omega(v_A)$ solves $\xi(v_A, \omega(v_A)) = \omega(v_A)$.

In this environment, DDI/DEI depends on the *correlation* between the incumbent's and entrants signals, not on the degree of commonality in values. Intuitively, the good news about the quality of the good inherent in a higher k must overwhelm the bad news about the strength of competition in that k. With independent signals, $\pi'_A(v_B) < 0$ always holds. For a simple example where correlation in the signals leads to DDI, suppose the buyer has a value $v_B \sim F$, which he observes perfectly, while any entrant has a value $v_A = \theta v_B$, where θ is independent of v_B and has mean $\mathbb{E}[\theta] > 1$. An entrant's ex post payoff from a bidding war is $(v_A - v_B)_+ = (\theta - 1)_+ v_B$, so that, using the independence of v_B and θ, $\pi_A(v_B) = \mathbb{E}[(\theta - 1)_+] v_B$, which is strictly increasing in v_B.[11]

Example 2 (Adjudication: private politics and bargaining in the shadow of regulators)**.** The "buyer" is a firm with privately known profitability v_B, and the "seller" is a striking union that wants to negotiate

[9] The numerical bidding war/interruption examples in Table 1 below all have single peaked $\hat{P}$'s.

[10] Even though bidders here have asymmetric beliefs, the equilibrium of the symmetric full-information ascending auction model in Milgrom and Weber (1982)[p. 196] is an *ex post* Perfect Bayesian Nash Equilibrium, so it will remain an equilibrium for any beliefs of the bidders.

[11] The same distinctions can be drawn in a model where entrants know their value before making an entry decision. Assume the incumbent buyer's private observation is a signal ε_B that gives an estimate of his own value, $\varepsilon_A \sim G_\varepsilon$ denotes a generic entrant's signal of his (the entrant's) value, and $c \sim G_c$ denote his entry cost. The ε_A's and c's are independent of each other and across entrants. Let $\tilde{\pi}_A(\varepsilon_A, k)$ denote the entrant's bidding war expected payoff, as a function of ε_A, when he believes $\varepsilon_B \leq k$. Assuming, as is natural, that $\tilde{\pi}_A$ is increasing in the entrant's signalε_A, let $\bar{\varepsilon}(k,c)$ be the unique value satisfying $\tilde{\pi}_A(\bar{\varepsilon}(k,c), k) = c$. The probability of entry at state k is then $\Lambda(k) = \ell_0 \int_{\varepsilon_A \geq \bar{\varepsilon}(k,c)} dG_c(c) dG_\varepsilon(\varepsilon_A)$. The analysis can proceed as before, with the only change being that the DEI/DDI condition, while remaining the same in spirit, will be formulated more abstractly. The environment is DEI (DDI) if $\frac{\partial}{\partial k}\bar{\varepsilon}(k,c) > (<) 0$. Intuitively, if $\frac{\partial}{\partial k}\bar{\varepsilon} < 0$, then whenever the incumbent has a better signal (higher k), the entrant is *more* willing to enter (requires a lower minimal signal to enter), and the environment is DDI.

the highest possible wage. A crisis (spooked investors, public relations disaster, quality collapse due to the ongoing strike) that threatens to wipe out the firm arrives at rate ℓ_0. The audience is a regulator who, at the moment of crisis, must decide on a (possibly inefficient) surplus split. Hence, $(\pi_S^d(v_B), \pi_B^d(v_B) := (\alpha^d v_B, \beta^d v_B), d \in \{Y, N\})$. where $(\alpha^d, \beta^d) \in (0,1] \times [0,1]$ are fixed parameters satisfying $\alpha^d + \beta^d \leq 1$ and $\alpha^Y > \alpha^N$, i.e., the union prefers $d = Y$. The regulator's payoffs $\pi_A^d(v_B)$ can be arbitrary subject to Assumption 3; the environment is DEI (DDI) if $\pi_A^Y - \pi_A^N$ is decreasing (increasing) so that the regulator is more likely to intervene in the union's favor when profitability is low (high).

3.1 Equilibrium Concept

Informal Summary: I focus on equilibria that are Markovian in the belief cutoff k. In the public game, the buyer chooses a reservation price strategy $P^{pu}(\cdot)$, such that v_B accepts the first time a price lower than $P^{pu}(v_B)$ is offered. Meanwhile, taking $P^{pu}(\cdot)$ as an (endogenous) inverse demand curve, the seller makes offers in "quantity space": for any initial state k, she chooses a path of cutoffs $t \mapsto K_t^k$ that results in continuation offers $t \mapsto P^{pu}(K_t^k)$.[12] The audience chooses an instantaneous rate $\mathcal{L}(\cdot)$ of taking the costly decision $d = Y$ as a function of the state. The additional twist with private offers is that the audience does not observe the offers, and so it must form time-dependent beliefs on the buyer's type. The game then has a two-dimensional state (cutoff k and calendar time t) that both buyer and seller condition on, and one has an additional fixed point on beliefs: along the path of play, the audience's beliefs about the types of the buyer who have been ruled out must be correct.

The set of all possible non-increasing cutoff paths is unmanageably large as a strategy space for the seller. To make the analysis tractable, I consider a restriction on this space that still allows for rich dynamics:

Definition 4 (Seller Strategy Space)**.** An action plan for the seller is a cutoff path $t \mapsto K_t$, with generic value k. $\mathbf{K}$ (in boldface) denotes the entire path $\{K_t\}_{t\geq 0}$. A cutoff path $\mathbf{K}$ is *admissible* if it is (i) measurable, (ii) non-increasing, (iii) right-continuous, and (iv) has no singular-continuous parts. Let $\mathcal{A}_k$ denote the set of admissible paths with initial value k, i.e, that satisfy $K_{0^-} = k$.

Definition 5 (Public Offers Equilibrium)**.** Take a triple of policies

$$\Sigma = \Big(\{\mathbf{K}^k\}_{k\in[0,1]}, P^{pu}(\cdot), \mathcal{L}(\cdot)\Big)$$

together with a value function for the seller $J_S^{pu}(\cdot)$.[13] Denote by $\mathbf{P}(\mathbf{Q}) = \{P^{pu}(Q_t)\}_{t\geq 0}$ and $\mathbf{L}(\mathbf{Q}) = \{\mathcal{L}(Q_t)\}_{t\geq 0}$ the prices and costly intervention rates that Σ prescribes in response to an admissible cutoff path $\mathbf{Q}$. Then Σ is a **Markov Equilibrium** of the public game if:

1. **Buyer Optimality** For all $v_B \in [0,1]$, $k \in [0,1]$, accepting at $\tau^* = \inf\{t : P(K_t^k) \leq P(v_B)\}$ solves the buyer continuation problem

[12] In the main text, to simplify the presentation, I assume the seller cannot mix. The extensions in appendices, which cover the gap case and more general primitives, require mixing, which is handled with a different formulation of the seller strategy space. The no-mixing equilibria in the main text are robust to giving the seller that richer strategy space. Moreover, note that a key result in the discrete-time literature on the Coasian bargaining is that there is no randomization along the path of Stationary (Weak Markov) Equilibria, and no randomization on or off path for Stationary Strong Markov Equilibria.

[13] I add the (slightly redundant) seller's induced value function to the set of equilibrium objects for convenience in stating a restriction later on.

$$\sup_{\tau\geq 0} \mathcal{U}_B^{v_B}(\tau; \mathbf{P}(\mathbf{K}^k), \mathbf{L}(\mathbf{K}^k))$$

2. Seller Optimality For any $k \in [0,1]$, $\mathbf{K}^k$ solves problem

$$\sup_{\mathbf{Q}\in\mathcal{A}_k} F_B(k)^{-1}\mathcal{U}_S(\mathbf{Q}, \mathbf{P}(\mathbf{Q}), \mathbf{L}(\mathbf{Q}))$$

and delivers continuation value $J_S(k)$.

3. Audience Optimality For all k, $\mathcal{L}(k) = \Lambda(k) = \ell_0 G(\Pi_A(k))$.

Private offers equilibria also consist of triples of reservation prices, cutoff paths, and costly intervention rates (together with an induced value function for the seller). Below, $\mathbf{K}^{k,t}$ denotes a cutoff path $s \mapsto K_{t+s}^{k,t}, s \geq 0^-$ satisfying the initial condition[14] $K_{t^-}^{k,t} = k$, and $\boldsymbol{\lambda}^* = \{\lambda_t^*\}$ is a path of costly intervention rates.

Definition 6 (Private Offers Equilibrium)**.** Take a triple of policies

$$\Sigma = \left(\{\mathbf{K}^{k,t}\}_{k\in[0,1],t\geq 0}, P^{pr}(\cdot,\cdot), \boldsymbol{\lambda}^*\right)$$

and a value function for the seller $J_S^{pr}(\cdot,\cdot)$. Denote by $\mathbf{P}_t(\mathbf{Q}) = \{P^{pr}(Q_s, t+s)\}_{s\geq 0^-}$ and $\mathbf{L}_t^* = \{\lambda_{t+s}^*\}_{s\geq 0^-}$ the prices and costly intervention rates that Σ prescribes for time t onwards in response to a cutoff path $\mathbf{Q}$. Then Σ is a Markov Equilibrium of the private game if

1. Buyer Optimality For all $k \in [0,1]$, $t \geq 0^-$, accepting at $\tau^* = \inf\{s : P^{pr}(K_{t+s}^{k,t}, t+s) \leq P^{pr}(v_B, t+s)\}$ solves v_B's continuation problem

$$\sup_{\tau\geq 0} \mathcal{U}_B^{v_B}(\tau, \mathbf{P}_t(\mathbf{K}^{k,t}), \mathbf{L}_t^*). \tag{5}$$

2. Seller Optimality For any $k \in [0,1]$, $t \geq 0^-$, $\mathbf{K}^{k,t}$ solves the seller's continuation problem

$$\sup_{\mathbf{Q}\in\mathcal{A}_k} F_B(k)^{-1}\mathcal{U}_S(\mathbf{Q}, \mathbf{P}_t(\mathbf{Q}), \mathbf{L}_t^*) \tag{6}$$

and delivers continuation value $J^{pr}(k,t)$.

3. Audience Optimality, Correct On-Path Beliefs For each $s \geq 0$, λ_s^* satisfies

$$\lambda_s^* = \Lambda\left(K_s^{1,0}\right) = \ell_0 G\left(\Pi_A\left(K_s^{1,0}\right)\right), \tag{7}$$

i.e., the entrants have correct on path conjectures about cutoff paths.

The seller is not mixing, so the costly intervention rate at t must match what it would have been had the audience known the on-path cutoff. In addition, in Condition 2, the seller now takes the intervention decisions as fixed, since the audience cannot observe the offers.

I focus on a subset of Markov equilibria in which the seller alternates between periods of sufficiently gradual trade and a few periods with atoms of trade.

Definition 7. (Smooth Trade) For a seller plan $\mathbf{K} = \{K_t\}_{t\geq 0}$, time intervals $[\underline{t}, \bar{t})$ where $t \mapsto K_t$ is absolutely continuous are *smooth trade regions*. Letting $\dot{K}_t$ denote the (a.e.) derivative in such regions, $-\dot{K}_t$ is the *trading speed*. A special case of a smooth trade region is a *quiet period*, i.e., an interval $[\underline{t}, \bar{t})$ with $K_{\bar{t}^-} = K_{\underline{t}}$.

[14] I allow $s = 0^-$, since $\mathbf{K}^{k,t}$ may jump t, i.e. $K_t^{k,t} < K_{t^-}^{k,t}$.

Definition 8. A Markov Equilibrium of either the public or the private game is *regular* if (i) jumps are isolated; (ii) cutoff paths $\mathbf{K}$ are differentiable on the interior of smooth trade regions; (iii), $J_S^{pu}(\cdot)$ (or respectively, $J_S^{pr}(\cdot,\cdot)$) are C^1; P^{pu} are P^{pr} are continuous at any (k,t) with smooth trade, and (v), for any on-path state (k,t) of the private game, the seller's continuation strategy has *absorbing smooth trade*: if trade is smooth at (k,t), it is smooth at $(k,t'), t' \geq t$. [15]

Below I abbreviate Regular Markov Equilibria as "RME's" or, when it does not cause confusion, simply "equilibria"

Discussion of Equilibrium Notion: Holding fixed the audience's actions, the buyer and seller are playing a continuous-time game with observable actions, which makes typical Nash notions of equilibrium problematic (Simon and Stinchcombe, 1989). Similar to Ortner (2017) and Daley and Green (2020), I define an equilibrium notion that is tailored to my setting but still captures key features from fully game-theoretic, discrete-time analyses of Coasian bargaining (along the lines of Strong Markov Stationary Equilibria, e.g, Fudenberg et al. (1985); Gul et al. (1986); Ausubel et al. (2002)). Unlike Ortner (2017) and Daley and Green (2020), my model has continuous types and no driving Brownian motion, so certain smoothness properties of value functions that arise naturally in those models—and are key to the analysis—cannot be directly derived here without some additional restrictions on admissible strategies. One of the contributions is therefore to provide a set of natural restrictions that allow for a streamlined analysis via HJB's in models with continuous types (Definition 4 for sufficiency/verification, and Definition 8 for necessity).

The first three conditions on cutoff paths in Definition 4 are standard. Condition (iv) has more bite. Since $\mathbf{K}$ is monotone decreasing, it has a Lebesgue decomposition. This already implies that $\mathbf{K} = \mathbf{A}+\mathbf{M}+\mathbf{N}$, where $\mathbf{A}$ is absolutely continuous, $\mathbf{M}$ is a piecewise constant jump function, and $\mathbf{N}$ is a singular continuous function. Therefore, the additional content in Condition (iv) is that the continuous part of $\mathbf{K}$ is sufficiently smooth: unless there is an atom of trade, the buyer can actually *see* the price dropping gently over time, rather than $\mathbf{K}$ only moving in "flashes."[16]

4 Effects of Privacy: Main Results

I can now formally state the main result of the paper, i.e., the seller's strong preference for private bargaining.

Theorem 1. *Let (k,t) denote any state reached on path in a private offers equilibrium Σ^{pr}, and let Σ^{pu} denote any public offers equilibrium. Let $J^{pr}(k,t)$ and $J^{pu}(k)$ denote the seller's continuation payoffs at state (k,t) in the two equilibria, respectively.*

1. *If DEI holds, and Σ^{pu} prescribes smooth trade at k, then*

$$J^{pr}(k,t) > J^{pu}(k)$$

2. *If DDI holds,*

[15] The role of the restriction is to derive necessary properties of equilibria from (classical) analysis of the HJB's. Existence in this class is shown by construction. Note that equilibria must still be robust to seller deviations in the full class of admissible strategies.

[16] Singular dynamics typically arise when trying to keep a Brownian state above some reflecting boundary (e.g., Daley and Green (2012)).

$$J^{pr}(k,t) < J^{pu}(k).$$

Ignoring for a moment the smooth trade hypothesis in the first item, the theorem essentially says that the seller prefers publicity if disagreement discourages costly intervention, and prefers privacy if disagreement encourages costly intervention.

Theorem 1 leaves open the possibility that the seller might prefer public bargaining in a DEI environment if the equilibrium starts with a jump.[17] I explore this issue numerically below. Nevertheless, there is still a sense in which the seller has robust preferences for privacy in a DEI world: public bargaining is dominated for the seller by a regime that is *nearly* private. Consider a hybrid "starting-public" game, where the first offer is made public and all subsequent offers are private. Defining RME's for this hybrid game is notationally cumbersome, so I relegate the formal definition to Online Appendix A.1, but the main idea is straightforward. Since the audience observes the initial offer, the seller controls the initial level of costly intervention. After that first moment, players face a private game with initial cutoff equal to the seller's initial public choice. In DEI environments, this nearly private regime is uniformly better for the seller than public bargaining:

Theorem 2. *If disagreement encourages costly intervention, the seller strictly prefers every RME of the hybrid starting-public game to any RME of the public game.*

If there were some benefit to the initial jump of a public offers RME, the seller could always mimic that jump in the hybrid game with her initial offer. Since bargaining is private after the first instant of the hybrid game, the seller can avoid the harmful effects of trading smoothly in public that follow the initial jump of the public game.

The statements (and indeed the proofs) of Theorems 1 and 2 use very little of the structure of equilibria, so the channel through which privacy affects the seller's payoffs is not yet clear: when the seller prefers public offers, is this because prices are higher? Because delay is reduced? Both?

A full analysis of the buyer's interim preferences is intractable, but under a natural additional condition, I decompose the payoff effects of privacy into its constituent price and delay channels; that decomposition also pins down the buyer's ex ante preferences over privacy and highlights a fundamental disagreement between the two parties.

Definition 9. Say 'disagreement has diminishing effects (on intervention)' if $|D'(\cdot)|$ is weakly decreasing.

Since $D'(\cdot)$ has a constant sign, the diminishing effects condition is equivalent to $D(\cdot)$ being concave under DEI but convex under DDI; intuitively, the audience responds more strongly to what the negotiation reveals early in the game, when the state is high and there is much to learn, than late in the game when most of the uncertainty is resolved.

Theorem 3 (Price and Delay Rankings)**.** *Let $p^{pu}(k)$ and $p^{pr}(k)$ be the public and private prices, respectively, that $v_B = k$ pays when he buys the good before intervention. Let $\mathbf{K}^{pu}$ and $\mathbf{K}^{pr}$ denote the equilibrium cutoff paths. Then*

$$p^{pu}(k) \begin{cases} < p^{pr}(k)\, \forall k > 0, \ \textit{if DEI holds} \\ > p^{pr}(k)\, \forall k > 0, \ \textit{if DDI holds} \end{cases}$$

[17] Below I provide conditions that guarantee existence of public offers equilibria starting with smooth trade.

If, in addition, F_B is log-concave and disagreement has diminishing effects, then

$$K_t^{pu} \begin{cases} < K_t^{pr} \text{ for all } t > 0, \textit{if DEI holds} \\ > K_t^{pr} \text{ for all } t > 0, \textit{if DDI holds}. \end{cases} \tag{8}$$

Said differently, *the price channel dominates*—the seller prefers the informational regime with higher prices, regardless of the resultant delay. Insofar as the seller's ability to maintain high prices reflects her commitment power, the theorem shows that privacy enhances that commitment power under DEI but undermines it under DDI.

Log-concavity and diminishing effects assumptions ensure that the private equilibrium price is more (less) sensitive to the seller's beliefs when DEI (DDI) holds; this in turn is a sufficient condition under which speeds of trade can be ranked, since the more sensitive price is to the seller's belief, the more tempted type k is to pretend to be $k - dk$, and therefore the more delay there must be in equilibrium in order to induce truthful reporting by k. I provide additional details in Section 6. In the interim, for a quick example where disagreement has diminishing effects, suppose the distribution of audience costs G is uniform. Then, in adjudication games, disagreement has diminishing effects whenever the audience net payoffs π_A are concave under DEI and convex under DDI. The same is true for interruption games when π_A is *sufficiently* concave/convex as the case requires. In Example 1 (an interruption game with entry of a symmetric IPV bidder), one has DEI with diminishing effects of disagreement for the canonical case of $F_E = F_B = G = U[0, 1]$.

For the following result, the *bilateral surplus* refers to the sum of the buyer's and seller's utilities, i.e., the total surplus excluding the audience's utility.

Corollary 1 (Buyer Preferences, Bilateral Surplus)**.**

1. *If DEI holds, there exists $k' < 1$ such that $v_B \geq k'$ strictly prefer public bargaining; if DDI holds, there exists $k' < 1$ such that $v_B \geq k'$ strictly prefer private bargaining.*
2. *Suppose that F_B is log-concave, disagreement has diminishing effects, and*

$$\pi_B^d(k) + \pi_S^d(k) = k \quad \forall k, \forall d. \tag{9}$$

i.e., intervention neither adds to nor subtracts from the bilateral surplus enjoyed by the buyer and the seller. Then there is an ex ante conflict of interest *over the privacy of negotiation:*

   (a) If DEI holds, private bargaining strictly shrinks the ex ante bilateral surplus, and the buyer strictly prefers public bargaining ex ante.

   (b) If DDI holds, private bargaining strictly grows the ex ante bilateral surplus, and the buyer strictly prefers private bargaining ex ante.

   (c) Comparisons (a) and (b) also hold type by type: for any k, the expected discounted bilateral surplus for the buyer and seller conditional on $v_B = k$, in expectation over the audience's decisions, is lower (higher) under private bargaining when DEI (DDI) holds.

Condition (9) says that the bilateral surplus shared by the negotiating parties is the same with and without intervention. Intuitively, audience intervention on its own does not give the buyer and the seller any reason to accelerate or delay the agreement, since the pie they split between them is the same size regardless. The condition holds, for instance, in Examples 2 (adjudication with surplus split) if

$\alpha^d + \beta^d = 1, d \in \{Y, N\}$ and in Example 1 (interruption with bidding war) if the bidding war is an ascending auction with no reserve. [18]

Proof of Corollary 1. The preference ranking for high types follows by continuity: the highest type always trades before intervention in either the public or private games, and pays a lower price price in the public game under DEI and in the private game under DDI. Buyer payoffs are continuous in type (by standard mechanism design arguments), so this preference extends to sufficiently high types.

Items 2(a) and 2(b) follow from 2(c). To prove the latter, fix a buyer type $v_B = k$, let τ_k denote the time at which k trades in the absence of intervention and σ denote the (random) time at which an audience intervention occurs. Under Condition 9, the eventual surplus enjoyed by the buyer and seller is the same regardless of the audience's choices, so their expected bilateral surplus, conditional on k, is $\mathcal{S}(k) := k\mathbb{E}[e^{-r\tau_k \wedge \sigma}|k]$. Take the DDI case. By Theorem 3, τ_k is strictly greater for all $k > 0$ in the public game. The audience cannot condition on v_B; the audience costs c and the arrival of intervention opportunities are independent of v_B. Hence σ is independent of the true type k. In adjudication games, σ is also independent of the information regime (it occurs at a constant rate ℓ_0). In interruption games, under DDI, σ is stochastically greater (smaller) under public (private) offers, since $\lambda_s^* = \Lambda(K_s^{pr}) < \Lambda(K_s^{pu}) \forall s > 0$. Altogether, under DDI, $\mathcal{S}(k)$ is strictly smaller for all $k > 0$ under public offers. The reverse ranking holds under DEI. □

5 Main Argument

Next, I give the crux of the argument behind Theorem 1. There are two parts to the argument. First, I formalize the heuristic "speeches" given by the different players in the introduction: the seller's 'outside option'—what she gets from stalling indefinitely—is higher in the private game under DEI but lower in the private game under DDI. Second, I show that, when the seller screens smoothly, her payoffs exactly match this outside option.

Seller Outside Options (Ex Ante Value of Stalling) Suppose that, having reached state k at time t, the seller deviates and "stalls," i.e., makes unacceptable offers forever after.' The expected discount to costly intervention that would result from this is $D(k)$ in the public game and $D^X(t|\boldsymbol{\lambda}^*)$ in the private game, and the payoffs from this deviation are

$$\begin{aligned} R^{pu}(k) &:= D(k)\Pi_S(k) + \frac{\ell_0}{\ell_0 + r}\Pi_S^N(k), && \text{if public} \\ R^{pr}(k, t|\boldsymbol{\lambda}^*) &:= D^X(t|\boldsymbol{\lambda}^*)\Pi_S(k) + \frac{\ell_0}{\ell_0 + r}\Pi_S^N(k) && \text{if private} \end{aligned} \tag{10}$$

In the public game, the audience can see the seller stalling, so its belief cutoff remains fixed at k. Meanwhile, in the private game, the audience would never observe the seller's deviation, which is why the discount to costly intervention is computed using the equilibrium rate. The seller's equilibrium payoffs in the public and private game are at least R^{pu} and R^{pr}, respectively, since the seller could

[18] There is an inherent difficulty in generalizing Theorem 1 to either (i) a uniform statement about buyers' interim preferences over regimes, or (ii) a statement about efficiency. Even for a buyer being given a static choice between facing an auction or facing an seller-optimal posted price, whether the buyer prefers the second scenario depends on fine details of the distribution of his and his opponent's types. By the same token, it is not clear in my model whether a given buyer prefers to shift discounted probability mass towards or away from the intervention scenario.

always guarantee that value for herself by refusing to trade indefinitely.[19]

Now the key fact: if (k,t) is on the path of the private game, the ranking of the outside options $R^{pr}(k,t)$ and $R^{pu}(k)$ is completely determined by whether DEI or DDI holds. First, with private offers, the audience has correct beliefs on path, so at time t it correctly infers that the current cutoff is k and intervenes at a rate $\lambda_t^* = \Lambda(k)$. Second, starting from an initial level of $\Lambda(k)$, the costly intervention rate$\{\lambda_{t+s}^*\}_{s\geq 0}$ *rises over time if DEI holds and falls if DDI holds*: the audience's beliefs about the buyer's type drop, which leads to more costly intervention in the former case and less in the latter— this is the formal counterpart to the heuristic speeches in the introduction. For all (k,t) on the path of the private game,

$$\begin{aligned} &\text{DDI holds} \Rightarrow D(k) \geq D^X(t|\boldsymbol{\lambda}^*) \Rightarrow R^{pu}(k) \geq R^{pr}(k,t|\boldsymbol{\lambda}^*), \\ &\text{DEI holds} \Rightarrow D(k) \leq D^X(t|\boldsymbol{\lambda}^*) \Rightarrow R^{pu}(k) \leq R^{pr}(k,t|\boldsymbol{\lambda}^*). \end{aligned}, \tag{11}$$

i.e., the seller's outside option is therefore higher or lower in the public game depending on whether DDI or DEI holds. The second key step links those outside options to the equilibrium payoffs:

Lemma 1 (Seller's Payoffs = Ex Ante Value of Stalling)**.** *For any RME's of the private and public games,*

1. *At any k such that the seller trades smoothly in the public game,*

$$J_S^{pu}(k) = R^{pu}(k)$$

2. *At any on-path (k,t) of the private offers RME at which smooth trade is prescribed,*

$$J_S^{pr}(k,t) = R^{pr}(k,t|\boldsymbol{\lambda}^*)$$

I prove the lemma for the interruption case, where the notation is simpler. Under public offers, having reached a state k with smooth trade, the seller's value function satisfies the HJB

$$\begin{aligned} rJ_S^{pu}(k) = \sup_{|\dot{k}|\in[0,\infty)} \Bigg\{ &(P^{pu}(k) - J_S^{pu}(k))\,\frac{f_B(k)}{F_B(k)}|\dot{k}| + \Lambda(k)[\Pi_S(k) - J_S^{pu}(k)] \\ &+ (J_S^{pu})'(k)(-|\dot{k}|) \Bigg\} \end{aligned} \tag{12}$$

where $|\dot{k}|$ denotes the seller's choice of trading speed. The first term corresponds to "flow" profits from the marginal types who accept the seller's offer. The seller collects $P^{pu}(k)$ from the "number" of marginal types $f_B(k)/F_B(k)$, multiplied by the speed $|\dot{k}|$ at which she is moving through the margin; upon acceptance, the game ends, so the seller also loses the continuation value $J_S^{pu}(k)$. The second term is the flow payoff from the event of costly intervention after a rejection (in which case the game ends, and the seller also "loses" $J_S^{pu}(k)$). The third term is the continuation payoff from moving to a new state.

The seller can choose the current trading speed $|\dot{k}| \in \mathbb{R}_+$ freely, and this variable enters the right hand side of (12) linearly. Therefore, for (12) to hold, either (i), the coefficients on $|\dot{k}|$ equal exactly 0, and the seller is indifferent over all speeds, or (ii), the coefficients add up to something strictly negative, and $|\dot{k}| = 0$ is uniquely optimal. In either case, $|\dot{k}|$ drops out from (12), which yields

[19] In the private game, the value the seller can guarantee for herself at (k,t) by stalling depends on the audience's conjecture has about future play).

$$J_S^{pu}(k) = \frac{\Lambda(k)}{\Lambda(k)+r}\Pi_S(k) = D(k)\Pi_S(k) = R^{pu}(k) \tag{13}$$

When offers are private, the audience's behavior depends on calendar time, which makes the seller's problem non-stationary. One then requires some restrictions on jumping behavior to pin down her continuation payoffs—but the basic idea, relying on the linearity of the HJB in the trading speed, still applies. Let $\boldsymbol{\lambda}^* = (\lambda_s^*)_{s\geq 0^-}$ denote the equilibrium costly intervention rate with private offers. Then, at a state (k,t) with smooth trade, the seller's value function must satisfy the HJB

$$rJ_S^{pr}(k,t) = \sup_{|\dot{k}|\in[0,\infty)} \Bigg\{ \left(P^{pr}(k,t) - J_S^{pr}(k,t)\right)\frac{f_B(k)}{F_B(k)}|\dot{k}| + \overbrace{\lambda_t^*[\Pi_S(k) - J_S^{pr}(k,t)]}^{\text{rejection \& costly intervention}} \\ + \underbrace{\frac{\partial}{\partial k}J_S^{pr}(k,t)(-|\dot{k}|) + \frac{\partial}{\partial t}J_S^{pr}(k,t)}_{\text{game continues with new cutoff/time state}} \Bigg\}, \tag{14}$$

so if $|\dot{k}| < \infty$ solves (40),

$$rJ_S^{pr}(k,t) = \lambda_t^*[\Pi_S(k) - J_S^{pr}(k,t)] + \frac{\partial}{\partial t}J_S^{pr}(k,t), \tag{15}$$

which, in a regular equilibrium can be solved to yield[20]

$$J_S^{pr}(k,t) = \left[\int_t^\infty \lambda_s^* e^{-\int_t^s(\lambda_\nu^*+r)d\nu}ds\right]\Pi_S(k) = R^{pr}(k,t|\boldsymbol{\lambda}^*), \tag{16}$$

thereby proving the lemma.

The above discussion establishes a weak version of Theorem 1 for the case where both the public and private game have smooth trade. The final step behind Theorem 1 is to characterize the equilibria of the two games. The next result summarizes the parts that are germane to Theorem 1 (see Theorem 5 in the Appendix for details)

Theorem 4. *Equilibria for the public and private games exist and have unique on-path outcomes.*[21]

1. *In the unique private equilibrium outcome, the equilibrium cutoff path $\mathbf{K}^{pr}$ is strictly decreasing (there are no quiet periods) and continuous (trade is always smooth). In particular, the inequalities in* (11) *are strict.*
2. *Let $k^\dagger$ be as defined in Assumption 4. In the unique public equilibrium outcome, if $k^\dagger = 1$, there is smooth trade at all states. If $k^\dagger < 1$, then for $k > k^\dagger$, the cutoff immediately jumps down to $k^\dagger$ (there is an atom of trade); for $k \leq k^\dagger$, trade is smooth.*

[20] In a regular equilibrium, the ODE holds for (k,s), $s > t$. Multiplying the time $s > t$ equation by $e^{-\int_t^s(\lambda_\nu^*+r)d\nu}ds$ on both sides, and integrating with respect to s, backwards from $T > t$,

$$J^{pr}(k,t) = \int_t^T \lambda_s^* e^{-\int_t^s(\lambda_\nu^*+r)d\nu}ds\Pi_S(k) + e^{-\int_t^T(\lambda_\nu^*+r)d\nu}J^{pr}(k,T).$$

J^{pr} is bounded, so taking $T\to\infty$ yields (16).

[21] The crux of the existence proof is (i) for either case, establishing a mixed rate-and-impulse-control verification theorem for the seller's problem; and (ii) for the private case, establishing the existence of a fixed point of seller cutoff paths and entrant beliefs. This is not a trivial requirement, since the fixed point is characterized by a non-linear functional equation: the trading speed depends on the expected discount to costly intervention, which itself depends on the entire future history of costly intervention rates.

6 Ranking Prices and Delay

The price ranking result follows almost directly from the argument in Section 5. Suppose that type k trades smoothly in both the public and private offers equilibria (the full proof takes care of the case where k trades in a jump under public offers). The expressions for J^{pu} and J^{pr} pinpoint the unique price that makes the seller willing to trade smoothly at a positive speed, i.e., that which makes the coefficient on $|\dot{k}|$ in the seller HJB equal to 0:

$$\begin{aligned} \hat{P}^{pu}(k) &:= D(k)\pi_S(k) + \frac{\ell_0}{\ell_0 + r}\pi_S^N(k) + \overbrace{D'(k)\Pi_S(k)\frac{F_B(k)}{f_B(k)}}^{=\rho(k)} \\ \hat{P}^{pr}(k,t) &:= D^X(t|\boldsymbol{\lambda}^*)\pi_S(k) + \frac{\ell_0}{\ell_0 + r}\pi_S^N(k) \end{aligned} \tag{17}$$

where I have used $\frac{\partial}{\partial k}(F_B(k)\Pi_S(k)) = f_B(k)\pi_S(k)$ to simplify. At the time t_k at which k trades in the private game, the difference in the prices paid by k is

$$P^{pu}(k) - P^{pr}(k, t_k) = \left[D(k)\pi_S(k) - D^X(t_k|\boldsymbol{\lambda}^*)\pi_S(k)\right] + D'(k)\Pi_S(k)\frac{F_B(k)}{f_B(k)} \tag{18}$$

If DDI holds, $D'(k) > 0$ and $D(k) > D^X(t_k|\boldsymbol{\lambda}^*)$, and both terms in (18) are positive; if DEI holds, the opposite is true.

The term $\rho(k) := D'(k)\Pi_S(k)\frac{F_B(k)}{f_B(k)}$ in the public price is a measure of how much the seller ex post "regrets" (or "rejoices from", depending on DDI/DEI) trading at state k, given what she learns when her offer gets accepted. At a state k with smooth trade, only the marginal type $v_B = k$ accepts the equilibrium price offer. Hence, if the offer is accepted, the seller concludes *ex post* that $v_B = k$, and that she would have received $D(k)\pi_S(k) + \frac{\ell_0}{\ell_0+r}\pi_S^N(k)$ in expectation if only she had waited for intervention. When the price is lower than that ex post payoff, trading is "bad news" for the seller.

The role of log-concavity and diminishing effects of disagreement I discuss the role played by the log-concavity and diminishing-effects assumptions in ranking delay across the two games. To fix ideas, focus on the interruption DEI case, where $\rho(k) < 0$, i.e., the seller suffers ex post regret in the public game.

In continuous time, the speed of trade at a state k depends on the balance of two forces: the size of the gains from trade (trade is faster when the (endogenous) gains from trade are higher at a state), and the price impact from marginally changing the state (if the price impact is higher, trade is slower). Intuitively, large gains from trade means players are more motivated to avoid discounting and trade happens more quickly; a large price impact gives buyer k strong incentives to pretend to be $k - dk$, so trade must be slower in equilibrium to induce k to truthfully report its type.

The 'gains from trade'—and indeed most terms in the marginal price impacts $\frac{\partial}{\partial k}\hat{P}^{pu}(k)(-dk)$ and $\frac{\partial}{\partial k}\hat{P}^{pr}(k,t)(-dk)$—can be ranked in the desired direction directly from the price expressions above without the additional assumptions, e.g., gains from trade are larger in the public game.[22] Log-concavity of F_B and the diminishing effects condition come in as sufficient conditions to control the derivate of regret $\rho'(\cdot)$ that, from (17), will show up in the comparison of marginal price impacts. If disagreement

[22] Since P^{pu} and P^{pr} are the highest price at which the seller is willing to trade in the two games. The buyer's reservation price has an additional term $\frac{\partial}{\partial t}D^X(t|\boldsymbol{\lambda}^*)\pi_S(k)$ in the private game reflecting the time trend in prices for a given cutoff, but the term does not override the effect of the price ranking.

has diminishing effects, intervention against inframarginal types is less responsive over time. Since, intuitively, the seller is only willing to suffer regret trading with the marginal type in order to induce faster intervention against inframarginal types, we should expect that the diminishing effects condition should help the regret term to shrink in magnitude as k falls, i.e., $\rho'(k) < 0$ so that $\rho(k) < \rho(k - dk) < 0$. The log-concavity of F_B preempts a possible caveat to this reasoning—that the regret incentive also depends on the relative sizes of marginal types $\approx f_B(k)dk$ and inframarginal types $F_B(k)$. If the inframarginal types grow relatively faster than the marginal ones as the state drops (i.e., log-concavity is violated), then the regret incentive could *increase* over time even as the audience response weakens. Hence, log concavity and the diminishing effects condition together guarantee that $\rho'(k) < 0$, from which $\frac{\partial}{\partial k}\hat{P}^{pu}(k) < \frac{\partial}{\partial k}\hat{P}^{pr}(k,t)$ follows.

Remark 1 (Connection to Price Transparency Literature)**.** Theorem 3 contrasts with previous studies on the effects of transparency on the division of surplus (Fuchs et al., 2016; Kaya and Liu, 2015). These papers have typically found that making offers private (i) can be a Pareto improvement and will generally benefit the informed party, and (ii) makes prices more costly to the informed party. In Kaya and Liu (2015) and Fuchs et al. (2016), making offers private lowers prices by removing the *buyer*'s incentives to reject prices in order to signal his value to future sellers. In my model, making offers private removes the seller's incentives to manipulate costly intervention rates through her offers, but this removal may raise prices or it may lower them, depending on the audience's preferences.

Remark 2 (Ex Post Regret and Belief Manipulation)**.** In a model of bargaining in a lemons market, Daley and Green (2020) found that the uninformed party also made "ex post regrettable" offers. The forces, however, are very different. In their model, the uninformed buyer offers prices that lose money conditional on the "bad" seller –and are only taken by the bad seller— as a way to experiment and learn: when the offer is rejected, the buyer becomes more certain that the type is high, which is useful information going forward. The seller in my model has no experimentation motive. She exposes herself to ex post regret/rejoicing because she wants to manipulate the audience's beliefs about market conditions which she herself does not know. The seller's behavior thus bears a resemblance to the signal-jamming models of Riordan (1985) and Fudenberg and Tirole (1986). In those models, an incumbent firm that is uncertain about the intercept of a demand curve, say, privately sets a low price in the hopes that a competitor will become pessimistic about that intercept.

7 Discussion, Extensions

Summary The effect of privacy on bargaining depends on whether, on net, strategic audiences prefer intervening against high or low types of the informed party. If disagreement encourages costly intervention, publicizing offers will tend to hurt the uninformed player making offers, moving prices against her, so that there is a trade-off between the ability to manipulate beliefs that comes with public offers, and the way this ability exacerbates the seller's commitment problem. If it *discourages* costly intervention, publicizing offers will tend to help the uninformed player making offers, moving prices in her favor. Publicity forces a new strategic calculus on the seller: her offers to the marginal type, by speeding up or slowing down the eventual interruption of negotiations, can affect the profits she receives from *inframarginal* types.

λ, α	0.3	0.7	1	3	5
0.3r	0.154125	0.211781	0.26673	0.420751	0.401806
0.7r	-0.151129	0.0723935	0.1383	0.26725	0.295848
r	-0.279311	-0.0262746	0.0479	0.191389	0.218542
3r	-0.986127	-0.55731	-0.4452	-0.266756	-0.236942
5r	-1.55012	-0.940307	-0.8066	-0.625042	-0.607774

Table 1: Proportional gain from private offers $\frac{J^{pr}(k=1,t=0)-J^{pu}(k=1)}{J^{pr}(k=1,t=0)}$, initial public jumps, DEI. $r=1$. Interruption model, audience is entrant, ascending auction. $F_B(v)=F_E(v)=v^{\alpha}$.

Benefits of an initial jump under public offers Theorems 1 and 2 highlight one rationale for bargaining publicly: if continued disagreement would weaken the seller's hand by delaying costly intervention, then the fear of that weakening hand gives the seller a kind of commitment power. But in the gap between Theorems 1 and 2, there lies a very different rationale, which I now explore numerically. When DEI holds and $\hat{P}^{pu}$ is decreasing at $k=1$, the public game starts with a jump, and Theorem 1 does not rank public and private bargaining for the seller. Even though the seller's continuation value after the jump will be lower than what it would have been at the equivalent state of the private game, the seller may still be better off net of the jump, for two reasons. First, by trading instantly with some high valuation types, the seller economizes on any discounting costs that she would have incurred had she traded with them gradually. Second, by jumping the state, the seller can "jumpstart" costly intervention, discontinuously accelerating, at the very beginning of the game, the rate at which the audience will intervene in the future.

Table 1 investigates whether these two salutary effects of public offer jumps can overwhelm the inherent advantages of private bargaining in DEI environments. I compute the proportional loss from private bargaining, $\frac{J^{pr}(k=1,t=0)-J^{pu}(k=1)}{J^{pr}(k=1,t=0)}$ for the case where costly intervention consists of entry by a competing buyer, followed by an ascending auction. I assume for the simulation that the entrant value v_E is also drawn from the incumbent's distribution F_B, and $F_B(v)=v^{\alpha}$. The entrants' cost is drawn $c\sim U[0,1]$. Different columns of Table 1 correspond to different values of $\alpha\in\{0.3,0.7,1,3,5\}$, while different columns correspond to different values of ℓ_0, where $\ell_0\in\{0.3r,0.7r,r,3r,5r\}$. r is held fixed at 1. For each of these examples, $\hat{P}^{pu}$ is single-peaked, with an interior maximum $k^{\dagger}$; the public game thus starts with a jump, followed by smooth trade, and gives the seller a payoff

$$J^{pu}(k=1)=(1-F_B(k^{\dagger}))\hat{P}^{pu}(k^{\dagger})+F_B(k^{\dagger})R^{pu}(k^{\dagger})$$

while the private game yields $J^{pr}(k=1,t=0)=R^{pr}(1,0|\boldsymbol{\lambda}^{*})$.

The key feature of Table 1 is its diagonal orientation: values above the diagonal are almost all positive, and they grow as one moves up and to the right, while values below the diagonal are negative, and they become more negative as one moves down and to the left. In short, the initial jump in a public offers equilibrium can be beneficial when α is low, and it is more beneficial the higher ℓ_0 is.

The positive impact of low α's in the public game is intuitive. Notice that low α's (the left columns) correspond to very concave (and therefore weaker) CDF's for the distribution of buyer values; the high types that get jumped over in the initial burst of public-offers trade are therefore very rare. Screening them smoothly, as would be the case with private bargaining, becomes wasteful for the seller, since she endures delay in ruling out types that are ex ante very unlikely to exist. The magnitude of the effect, however, is modulated by ℓ_0, and only becomes quantitatively significant when ℓ_0 is high.

Comparison to the Static Case To highlight the effect of the lack of commitment, I briefly compare the results to the static/one-shot case. Consider a version of the interruption game where (i) the seller makes a single offer, (ii) if that offer is rejected, the audience can intervene with some probability, and then (iii) if no audience intervenes, both buyer and seller get zero. In a pure-strategy equilibrium of the private game, entrants will have correct beliefs about the offer and enter as though they knew the true offer. The seller could always induce this private offers outcome under public offers, or she could reoptimize further and choose a different offer. Therefore she would always prefers bargaining in public, strictly so if the public offers outcome differs from the private one.

In light of this heuristic argument, the seller's extreme lack of commitment changes the effect of privacy dramatically: it makes the preferences for publicity vs privacy *contingent on the information structure* (nothing about the "mimicking" argument that made the seller prefer public offers in the static case depended on the valuation structure of buyer and entrants), and it creates induced preferences for privacy that are impossible in static settings, such as a strict preference for privacy.

Extensions The appendices extend the model in a few different directions. Online Appendices B and A.4 relax Assumption 3.1 (on the shape of $\hat{P}^{pu}$) and 3.2 (on the no-gap condition) respectively. Relaxing Assumption 3.1 complicates the dynamics in the public game, but the payoff comparisons are mostly unchanged.

With a gap, late enough in the game the seller may prefer making a pooling offer and trading with all remaining types to her stalling payoff. If the seller were to make this pooling offer deterministically, the price would have to drop discontinuously; predicting this drop, the types who are supposed to trade right before the atom would have no incentive to do so. To maintain incentive compatibility for the buyer, the seller must therefore impose real delay—a quiet period or impasse during which the seller only makes non-serious offers. Achieving this behavior in a Markov way requires an expanded strategy set for the seller that allows her to make a pooling offer at a random stopping time, but the main results and all of the economics by and large go through identically: the central argument only relies on necessary conditions at times of smooth trade, and these necessary conditions are unaffected by the expanded strategy sets.

Equilibrium construction is substantially more involved, and in the private game, the verification approach from the no-gap case breaks down. The dynamics change, so that equilibria typically involve smooth trade, an impasse, and a burst of trade at the end. The resulting value functions have kinks and the flow payoffs–since price jumps down at some point–are discontinuous. I provide a complete treatment for the public game. For the private game, a full verification for the seller runs into technical issues near the frontier of the control literature, i.e. regarding the uniqueness of viscosity solutions for discontinuous HJB's. The economics, however, are clear enough. The "right" construction, in terms of incentives that would support private on-path play similar to the public gap case, is as follows. The possibility of making a profitable pooling offer creates a region in (t, k) space inside which it is overwhelmingly tempting to make a pooling offer. The boundary of this region usually *cannot be crossed on path*, since prices would drop discontinuously and deterministically. Hence, in equilibrium the cutoff path must "reflect" off the boundary, which is achieved by the seller mixing between a "normal" offer and a discontinuously generous pooling offer, in such a way that the marginal type remains indifferent between accepting and rejecting and the cutoff path skirts the pooling region. Eventually, normal trade "stops," with the seller mixing between holding firm and making the pooling offer.

Appendix A.2 provides a construction for the DEI case that is (i) buyer and audience optimal; (ii) satisfies the consistent-beliefs fixed point condition; (iii) induces a value function for the seller that solves the HJB in a viscosity sense; and (iv) and is seller optimal starting at states from which the non-differentiable locus of the value is inaccessible. [23] Uniqueness of viscosity solution to the HJB in the seller's private game problem remains an open question.

Implications for Disclosure The results imply the equilibrium effects of "transparency" policies can vary dramatically depending on the market micro-structure. The SEC, for example, requires publicly traded companies to disclose information that is material to shareholders. Since serious offers by buyers are material, publicly traded companies are constrained in how privately they can negotiate acquisitions. This paper ignores the moral hazard dimension in mergers and acquisitions that typically motivates these rules, but it uncovers an unexpected trade-off. If the universe of acquirers consists mostly of competitors or upstream firms, whose values stem from their own synergies with the target's assets ('strategic' acquirers), then one should expect disagreement in bargaining to encourage entry. My results indicate a margin on which the shareholders of such a target are harmed by forcing management to bargain with an acquirer in full public view. Even then, a regulator may pursue this disclosure policy because of its countervailing benefit of lowering bargaining delays. If, on the contrary, the universe of acquirers consists mostly of private equity funds ('financial' acquirers), whose values stem from an eventual resale of the company after subjecting it to similar kinds of financial engineering, then one should expect disagreement in bargaining to *discourage* entry. Disclosure may then benefit the target's shareholders, at the social cost of increasing bargaining delays.

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[23] Standard viscosity uniqueness/verification results do not apply, since the HJB data is discontinuous, the policies spend positive time on the non-differentiable locus, and the locus can be accessed from above and from below.

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A Equilibrium Characterization

This section provides a complete characterization of public and private equilibria, as summarized in the following:

Theorem 5 (Equilibrium)**.** *Under Assumptions 1-4, the private and public games have unique RME's outcomes. The following is an equilibrium triple* $(\mathbf{K}^{pr}, P^{pr}(\cdot,\cdot), \boldsymbol{\lambda}^*)$ *of the private game,*

1. $P^{pr}(\cdot,\cdot) = \hat{P}^{pr}(\cdot,\cdot|\boldsymbol{\lambda}^*)$*, where* $\hat{P}^{pr}$ *is given by* (17).
2. *Trade happens smoothly (no jumps) at all states* (k,t)*, at a strictly positive speed (no quiet periods). Starting from a state* (k_0,t_0)*,* K_t^{pr,k_0,t_0} *evolves according to* $K_t^{pr,k_0,t_0} = k_0 - \int_{t_0}^t \varphi^{pr}(K_s^{pr,k_0,t_0}, s; \boldsymbol{\lambda}^*)ds$*, i.e.,* K_t *moves at speed* $\varphi^{pr}(K_t,t;\boldsymbol{\lambda}^*)$ *almost everywhere,*[24] *where*

$$\varphi^{pr}(k,t;\boldsymbol{\lambda}^*) := \left[\frac{\zeta^{pr}(t)(k-\hat{P}^{pr}(k,t)) + \frac{\partial}{\partial t}\hat{P}^{pr}(k,t) - \lambda_t^*\pi_B(k) - \ell_0\pi_B^N(k)}{\frac{\partial}{\partial k}\hat{P}^{pr}(k,t)}\right]$$

and $\zeta^{pr}(t) := -\frac{\partial}{\partial t}\log\delta(t|\boldsymbol{\lambda}^*)$ *denotes the rate at which intervention is discounted at time* t*.*

with the unique private RME outcome given by the above system starting from $k=1, t=0^-$*.*

In the equilibrium policy triple $\left(\mathbf{K}^{pu}, P^{pu}(\cdot), \mathcal{L}(\cdot)\right)$ *of the public game,*

1. $P^{pu}(k) = \begin{cases}\hat{P}^{pu}(k), k \le k^\dagger \\ \hat{P}^{pu}(k^\dagger), k > k^\dagger\end{cases}$ *where* $\hat{P}^{pu}$ *is given by* (17)
2. *For* $k > k^\dagger$*, the cutoff immediately jumps down to* $k^\dagger$*, while for* $k \le k^\dagger$*, trade happens smoothly at a strictly positive speed. For an initial state* $k_0 \le k^\dagger$*, the cutoff evolves according to*

$$K_t^{pu,k_0} = k_0 - \int_0^t \varphi^{pu}(K_s^{pu,k^0})ds.$$

where

$$\varphi^{pu}(k) := \zeta^{pu}(k)\left[\frac{k - \hat{P}^{pu}(k) - D(k)\pi_B(k) - \frac{\ell_0}{\ell_0+r}\pi_B^N(k)}{(\hat{P}^{pu})'(k)}\right] \tag{19}$$

and $\zeta^{pu}(k) = -\frac{\partial}{\partial t}\delta(t|\{\Lambda(k)\}_{s\ge 0})$ *denotes the rate at which intervention is discounted at state* k*.*

Remark 3. Below I mostly provide the results for the interruption case. The adjudication case goes through identically, simply by replacing the expressions for $D, D^X(t|\boldsymbol{\lambda}^*), \hat{P}^{pu}, \hat{P}^{pr}, R^{pu}, R^{pr}$ throughout with their adjudication counterparts, and replacing the expressions for the smooth trade speeds as described in Theorem 5.

First, I provide details on the skimming property.

Lemma 2. *Let* σ *be a random time of intervention and* $\mathbf{P} = \{P_t, t \ge 0\}$ *denote some (possibly random) sequence of price offers, and* $\mathbf{d} = \{d_t, t \ge 0\}$ *a sequence of audience intervention decisions. Under Assumption 2, if* v *finds it weakly optimal to accept* P_t *at a time* t *given a history for* σ, P*, and* d*, then any type* $v' \ge v$ *also finds it weakly optimal accept.*

[24] By Assumptions 3 and 2, the numerator in the speed is strictly positive. The denominator is strictly positive for all k if $\pi_S'(1) > 0$, and strictly positive for $k < 1$ and zero for $k = 1$ if $\pi_S'(1) = 0$. To accomodate that case, I consider solutions to the initial value problem $K_t = -\varphi^{pr}(K_t,t;\boldsymbol{\lambda}^*)$ in the Caratheodory sense (see Lemma 9).

Proof of Lemma 2. I prove the lemma for the adjudication case. The interruption case is the same but simpler. I start off with some helpful notation. Let $\{\mathcal{H}_t\}_{t\geq 0}$ be the σ-algebra generated by $\sigma < t, \{P_{t'}, t' \leq t\}, \{d_{t'}, t' \leq t\}$, and let

$$U(v,\tau) := \mathbb{E}\left[e^{-r\tau}\mathbf{1}_{\sigma\leq\tau}(v - P_\tau) + e^{-r\sigma}\mathbf{1}_{\sigma<\tau}[d_\tau\pi_B^Y(v) + (1-d_\tau)]\pi_B^N(v)|\mathcal{H}_t\right].$$

Now fix any v, v' with $v' > v$. Adding and subtracting $v', \pi_B^Y(v'), \pi_B^N(v')$, one obtains the following relations for any $\tau \in \mathcal{T}$:

$$\begin{aligned} U(v,\tau) &= U(v',\tau) + \mathbb{E}\Big[e^{-r\tau}\mathbf{1}_{\sigma\leq\tau}(v-v') \\ &\qquad + e^{-r\sigma}\mathbf{1}_{\sigma<\tau}[d_\tau(\pi_B^Y(v) - \pi_B^Y(v')) + (1-d_\tau)(\pi_B^N(v) - \pi_B^N(v'))\Big]. \\ &\geq U(v',\tau) + (v - v')\underbrace{\mathbb{E}[e^{-r\tau\wedge\sigma}]}_{:=\delta_\tau} \end{aligned} \tag{20}$$

where the inequality uses Assumption 2.

If v is indifferent between accepting and rejecting P_t, then using (20), for any $\tau \in \mathcal{T}$,

$$v - P_t = \sup_{\tilde\tau} U(v,\tilde\tau) \geq U(v',\tau) + (v-v')\delta_\tau \geq U(v',\tau) + (v - v')$$

Hence $v' - P_t \geq U(v',\tau)$ for arbitrary τ, and $v' - P_t \geq \sup_{\tilde\tau\in\mathcal{T}} U(v,\tilde\tau)$. □

Necessity

I prove that the private game has no jumps and no quiet periods for the interruption case; the remaining case follows as in Remark 3.

Lemma 3. *The private game equilibrium outcome has*

1. *No jumps in the cutoff path.*
2. *No quiet periods.*

Proof. Suppose, by way of contradiction, that a private offers RME features jumps. Say a given jump happens at time $\bar t$, and goes from $\bar k$ to $\underline{k} \in (\underline{v}, k)$. (The endpoint $\underline{k}$ must be strictly greater than $\underline{v}$, since seller can do better than that by making unreasonable offers indefinitely.) Since the equilibrium is regular, by Lemma 1 the seller's payoff at the end of the jump must satisfy

$$J_S^{pr}(\underline{k}, \bar t) = D^X(\bar t|\boldsymbol{\lambda}^*)\Pi_S(\underline{k}). \tag{21}$$

I now show the jump would never be optimal.[25] First, since there is smooth trade after the final jump, prices satisfy $P^{pr}(\underline{k}, \bar t) \leq \hat P^{pr}(\underline{k}, \bar t)$. Indeed, given the seller's continuation payoff in (21), $\hat P^{pr}(\underline{k}, \bar t)$ is the highest price at which $|\dot k| < \infty$ can solve the right hand side of the seller's HJB in (40) (since J_S^{pr} is C^1 in any RME, the HJB is necessary for seller optimality under smooth trade).

Let $A(k, k') = 1 - \frac{F_B(k')}{F_B(k)}$ for $k \geq k'$. Therefore, using (21) and the bound on P^{pr}, the seller's payoffs from executing the final jump are at most

[25] The logic of the argument borrows from Lemma 6 in Fuchs and Skrzypacz (2010).

$$
\begin{aligned}
&A(\bar{k},\underline{k})D^X(\bar{t}|\boldsymbol{\lambda}^*)\pi_S(\underline{k}) + (1-A(\bar{k},\underline{k}))D^X(\bar{t}|\boldsymbol{\lambda}^*)\Pi_S(\underline{k}) \\
&\qquad\qquad\qquad\qquad < \\
&A(\bar{k},\underline{k})D^X(\bar{t}|\boldsymbol{\lambda}^*)\mathbb{E}\left[\pi_S(v)|\underline{k}<v<\bar{k}\right] + (1-A(\bar{k},\underline{k}))D^X(\bar{t}|\boldsymbol{\lambda}^*)\Pi_S(\underline{k}) \\
&\qquad\qquad\qquad\qquad\qquad\qquad = D^X(\bar{t}|\boldsymbol{\lambda}^*)\Pi_S(\bar{k}),
\end{aligned}
\tag{22}
$$

The strict inequality follows from π_S being strictly increasing, and the equality follows from $A(\bar{k},\underline{k}) = \mathbb{P}(v_B \geq \underline{k}|v_B \leq \bar{k})$, $\Pi_S(\underline{k}) = \mathbb{E}[\pi_S(v_B)|v_B \leq \underline{k} < \bar{k}]$, and $\Pi_S(\bar{k}) = \mathbb{E}[\pi_S(v_B)|v_B \leq \bar{k}]$. The seller can always achieve the far right hand side of (22) by making unreasonable offers forever, so the jump cannot have been optimal. In particular, $J_S^{pr}(k,t) = D^X(t|\boldsymbol{\lambda}^*)\Pi_S(k)$ for all (k,t).

Next, suppose there were a quiet period on path in the private game. Concretely, suppose that, when type $v_B = k$ is marginal at time t along the equilibrium path, continuation play prescribes a quiet period until $t^* > t$. Let $V(k,t|\boldsymbol{\lambda}^*)$ be the continuation payoff to k from rejecting offers until t^*, given costly intervention rates $\boldsymbol{\lambda}^*$. If there is a quiet period at (k,t), rejecting must be weakly optimal for type k, so $P^{pr}(k,t) \geq k - V(k,t|\boldsymbol{\lambda}^*)$.

Then, since the quiet period is on path, and entrants have correct beliefs, $\lambda_z^* = \Lambda(k)$ for $z \in [t,t^*]$. In addition, as shown above, continuation play has no jumps; trading at (k,t^*) will therefore be smooth, at a price $\hat{P}^{pr}(k,t^*) = D^X(t^*|\boldsymbol{\lambda}^*)\pi_S(k)$. Hence,

$$
\begin{aligned}
V(k,t|\boldsymbol{\lambda}^*) &= \mathbb{E}_{k,t}^K\left[\mathbf{1}_{\sigma<t^*}e^{-r(\sigma-t)}\pi_B(k) + \mathbf{1}_{\sigma\geq t^*}e^{-r(t^*-t)}\left(k - D^X(t^*|\boldsymbol{\lambda}^*)\pi_S(k)\right)\right] \\
&= \frac{\Lambda(k)}{r+\Lambda(k)}\left[1-e^{-(r+\Lambda(k))(t^*-t))}\right]\pi_B(k) + e^{-(r+\Lambda(k))(t^*-t)}\left(k - D^X(t^*|\boldsymbol{\lambda}^*)\pi_S(k)\right)
\end{aligned}
\tag{23}
$$

(Recall that $\sigma|\sigma > t$ has pdf $z \mapsto \lambda_z^* e^{-\int_t^z \lambda_\nu^* d\nu}$). Plugging in $D(k)\pi_B(k) < k - D(k)\pi_S(k)$ (from Assumption 3) and

$$
D^X(t|\boldsymbol{\lambda}^*) = \frac{\Lambda(k)}{r+\Lambda(k)}\left[1-e^{-(r+\Lambda(k)(t^*-t))}\right] + e^{-(r+\Lambda(k))(t^*-t)}D^X(t^*|\boldsymbol{\lambda}^*),
$$

into (23) yields

$$
V(k,t|\boldsymbol{\lambda}^*) < k - D^X(t|\boldsymbol{\lambda}^*)\pi_S(k)
$$

Therefore, $P^{pr}(k,t) \geq k - V(k,t|\boldsymbol{\lambda}^*)$ yields $P^{pr}(k,t) > \hat{P}^{pr}(k,t)$. Once again, given $J_S^{pr}(k,t) = D^X(t|\boldsymbol{\lambda}^*)\Pi_S(k)$, $\hat{P}^{pr}(k,t)$ is the highest price at which $|\dot{k}| < \infty$ is still optimal for the seller. The quiet period is therefore strictly suboptimal. $\square$

Given the previous lemma, the strictness claims in Theorems 1- 4 and Corollary 1 follows from stochastic dominance.

Let σ^{pr} and σ^{pu} be the time of costly intervention conditional on state (k,t), *given the stalling deviation*. With DEI, given no quiet periods on the path of the private game, $\lambda_{t+s}^* > \Lambda(k)\forall s > 0$, $\lambda_t^* = \Lambda(k)$, so $\sigma^{pr}|\sigma^{pr} > t$ strictly first-order-stochastically dominates $\sigma^{pu}|\sigma^{pu} > t$. Since e^{-rt} is strictly decreasing in t, $D^X(t|\lambda^i) = \mathbb{E}[e^{-r\sigma^{pr}}|\sigma^{pr} > s]$ and $D(k) = \mathbb{E}[e^{-r\sigma^{pu}}|\sigma^{pu} > s]$, and the claim follows. The DDI case is symmetric.

Lemma 4 (Quiet Periods)**.** *There are no quiet periods in the public game.*

Proof. Take a cutoff path $\mathbf{K}^k$ by the seller with an initial quiet period of length t^*, i.e., $K_t^k - K_{0^-}^k = 0$ for $t \in [0,t^*)$ and $K_t^k > K_{t^*}^k$ for $t > t^*$. For $\Delta \in (0,t^*)$, type k's payoffs from waiting at $(k,0^-)$

satisfy

$$V(k)=\frac{\Lambda(k)}{r+\Lambda(k)}\left[1-e^{-(r+\Lambda(k))\Delta}\right]\pi_B(k)+e^{-(r+\Lambda(k))\Delta}V(k)\Rightarrow V(k)=D(k)\pi_B(k) \tag{24}$$

Rejection is weakly optimal for type k during the quiet period, so $P^{pu}(k)\geq k-D(k)\pi_B(k)$. Since trade is (trivially) smooth at k by hypothesis, from Lemma 1,$J_S^{pu}(k)=D(k)\Pi_S(k)$. Using $\frac{\partial}{\partial k}(\Pi_S(k)F_B(k))=\pi(k)f_B(k)$, the coefficients on $|\dot{k}|$ in the seller's HJB (12) therefore sum to at least $(k-D(k)(\pi_B(k)+\pi_S(k)))\frac{f_B(k)}{F_B(k)}-D'(k)\Pi_S(k)>0$ during a time interval of positive measure, where the inequality follows from Assumption 3. Hence, the seller's HJB, which is necessary given $J_S^{pu}\in C^1$, would be violated on a time interval of positive measure, a contradiction. $\square$

Next, I provide a useful lemma that characterizes the seller's incentives to jump the state when the price at the resulting state is given by $\hat{P}^{pu}$; additinoal lemmas connect the equilibrium price and $\hat{P}^{pu}$, which pins down the pattern of jumping in the game. In what follows, $F_B(k';k)=F_B(k')/F_B(k)$, and $f_B(k';k)=f_B(k')/F_B(k)$, for $k'\leq k$, denote the CDF and pdf of v_B conditional on a right-truncation at k. Let

$$L(k,k'):=\left[1-F_B(k';k)\right]\hat{P}^{pu}(k')+F_B(k;k')R^{pu}(k'),\quad k'<k \tag{25}$$

denote the payoff to the seller when she jumps the state from k to k' and resumes smooth trade when reaching state k'. Recall that R^{pu} denotes the seller's reservation value from stalling, which equals her payoffs from smooth trade.

Lemma 5. *Fix two states $0<\underline{k}<\bar{k}$. If*

$$\hat{P}^{pu}(\underline{k})\geq(\leq)\hat{P}^{pu}(k)\quad\forall k\in[\underline{k},\bar{k}], \tag{26}$$

then $L(\bar{k},\underline{k})\geq(\leq)\,R^{pu}(\bar{k})$, with strict inequality if (26) *is strict.*

Proof. Take some point $k^*\in[\underline{k},\bar{k}]$ at which $R^{pu}(k^*)\geq L(k^*,\underline{k})$. Rearranging that condition, $F_B(\underline{k};k^*)\left[\hat{P}^{pu}(\underline{k})-R^{pu}(\underline{k})\right]\geq\hat{P}^{pu}(\underline{k})-R^{pu}(k^*)$. Then

$$\begin{aligned}L_1(k^*,\underline{k})&=\frac{f_B(k^*)}{F_B(k^*)}F_B(\underline{k};k^*)\left[\hat{P}^{pu}(\underline{k})-R^{pu}(\underline{k})\right]\\&\geq\frac{f_B(k^*)}{F_B(k^*)}\left[\hat{P}^{pu}(\underline{k})-R^{pu}(k^*)\right]\\&\geq\frac{f_B(k^*)}{F_B(k^*)}\left[\hat{P}^{pu}(k^*)-R^{pu}(k^*)\right]=(R^{pu})'(k^*)\end{aligned}$$

The second inequality uses (26) and the last equality uses the definition of $\hat{P}^{pu}$. Since $L(\underline{k},\underline{k})=R^{pu}(\underline{k})$, the weak version of the lemma follows (e.g. Milgrom (2004), p. 124), and the strict one is proved similarly. $\square$

I can now pin down equilibrium prices in the public game.

Lemma 6. *Let **mon** $\hat{P}^{pu}(k)=\sup_{k'\leq k}\hat{P}^{pu}(k')$ be the monotone hull of $\hat{P}^{pu}$, i.e., the smallest increasing function that is everywhere above $\hat{P}^{pu}$. Then, in any public equilibrium,*

- *$P^{pu}\leq$ **mon** $\hat{P}^{pu}$.*
- *If Assumption 4.1 holds, $P^{pu}=$ **mon** $\hat{P}^{pu}$.*

Proof. First, I enumerate some key facts:

1. If equilibrium prescribes a jump between k and $k' < k$, P^{pu} must be constant in the interval $[k', k]$, and $P^{pu}(k) = \hat{P}^{pu}(k')$.
2. If equilibrium prescribes smooth trade at a positive speed in a neighborhood of k, then P^{pu}, $\hat{P}^{pu}$, and **mon** $\hat{P}^{pu}$ must equal each other in that neighborhood.
3. If **mon** $\hat{P}^{pu}$ is strictly increasing on a neighborhood of k, it must equal $\hat{P}^{pu}$ in that neighborhood.

To prove the first item in the lemma, notice that at any k with $P^{pu}(k) >$ **mon** $\hat{P}^{pu}(k)$, $P^{pu}(k) > \hat{P}^{pu}(k)$ as well, and smooth trade would not be incentive-compatible for the seller. Hence, equilibrium must involve a jump down to some $k' < k$. By the Fact 1 above,

$$P^{pu}(k) = P^{pu}(k') = \hat{P}^{pu}(k') \leq \textbf{mon}\ \hat{P}^{pu}(k),$$

a contradiction. It must then be the case that $P^{pu} \leq$ **mon** $\hat{P}^{pu}$.

To prove the second item in the lemma, let $\underline{k} = \inf\{k : P^{pu}(k) \neq \textbf{mon}\ \hat{P}^{pu}(k)\}$, and assume, to derive a contradiction, that $\underline{k} < 1$. Notice that **mon** $\hat{P}^{pu}$ is continuous, since $\hat{P}^{pu}$ is. Since $P^{pu} \leq$ **mon** $\hat{P}^{pu}$ and P^{pu} is weakly increasing, there must exist some $\varepsilon > 0$ such that **mon** $\hat{P}^{pu}$ is strictly increasing for $k \in (\underline{k}, \underline{k} + \varepsilon]$ and $P^{pu}(k) <$ **mon** $\hat{P}^{pu}(k)$ and $\hat{P}^{pu}(k) =$ **mon** $\hat{P}^{pu}(k)$ in that range. For future reference, note the additional consequence that $\bar{k} + \varepsilon$ must be strictly less that $k^\dagger$, where the latter is as defined in Assumption 4.

Jumps in equilibrium paths are isolated, so there must further exist some $\delta \in (0, \varepsilon)$ such that (i) equilibrium mandates a jump from $\underline{k} + \delta$ to $\underline{k}$, or (ii) equilibrium mandates smooth trade from $\underline{k} + \delta$ to $\underline{k}$. The second possibility is ruled out by the Facts 2 and 3 above: by Lemma 4, trade must occur at a strictly positive speed as the interval $(\underline{k}, \underline{k} + \delta)$ is traversed, which would require $P^{pu} = \hat{P}^{pu} =$ **mon** $\hat{P}^{pu}$ over that interval, a contradiction.

Suppose, then, that there were a jump between $\underline{k} + \delta$ and $\underline{k}$. It must then be the case that

$$\hat{P}^{pu}(\underline{k}) = \textbf{mon}\ \hat{P}^{pu}(\underline{k}) = P^{pu}(\underline{k}).$$

The first equality follows from $\underline{k} < k^\dagger$ and Assumption 4.1. The second equality follows from the continuity of **mon** $\hat{P}^{pu}$:

$$P^{pu}(k-) \leq P^{pu}(k+) \leq \textbf{mon}\ \hat{P}^{pu}(k+) = \textbf{mon}\ \hat{P}^{pu}(k+) = P^{pu}(k-)$$

From the last display, and the fact that jumps are isolated so smooth trade (possibly at zero speed) must resume at $\underline{k}$, the seller's payoffs from jumping to $\underline{k}$ from $\underline{k} + \delta$ must equal

$$L(\underline{k}, \underline{k} + \delta) := [1 - F_B(\underline{k}; \underline{k} + \delta)]\, \hat{P}^{pu}(\underline{k}) + F_B(\underline{k}; \underline{k} + \delta) D(\underline{k}) \Pi_S(\underline{k}).$$

However, $\hat{P}^{pu}$ is strictly increasing in $[\underline{k}, \underline{k} + \delta]$, so Lemma 5 implies that $L(\underline{k}, \underline{k} + \delta) < D(\underline{k})\Pi_S(\underline{k})$. The jump would be strictly suboptimal for the seller, a contradiction. Hence $\underline{k} = 1$, and $P^{pu} =$ **mon** $\hat{P}^{pu}$. □

Lemma 7 (Qualitative Dynamics)**.** *Consider any public equilibrium and any initial state k_0. Under Assumptions 1-4, if $k > k^\dagger$, the state jumps to $k^\dagger$ immediately. If $k \leq k^\dagger$ trade is smooth.*

Proof. From Lemma 6 and the fact that jumps are isolated, the seller's payoff an on-path jump from $k \leq k^\dagger$ to $\tilde{k} < k$ are $L(k, \tilde{k})$. Differentiating with respect to k and using (17), $L_1(k, \tilde{k}) =$

$A(\tilde{k}, k)(\hat{P}^{pu})'(\tilde{k}) > 0$. Jumping to $\tilde{k} \neq k$ is strictly suboptimal, and there must be smooth trade at $k \leq k^\dagger$. Finally, for $k > k^\dagger$, $\hat{P}^{pu}(k) < \hat{P}^{pu}(k^\dagger)$, so by Lemma 5, the seller strictly prefers jumping to $k^\dagger$ than trading smoothly. □

Uniqueness of Trading Speeds: Suppose that, given a price path $\mathbf{P} = \{P_s\}_{s\geq 0}$ and costly intervention rates $\boldsymbol{\lambda} = \{\lambda_s\}_{s\geq 0}$, a type v_B finds it optimal to trade at time t. Then, if $s \mapsto P_s$ is differentiable at t and $s \mapsto \lambda_s$ is continuous, the following first order condition is necessary:

$$\frac{\partial}{\partial t}\mathcal{U}_B^{v_B}(t, \mathbf{P}, \boldsymbol{\lambda}) = \frac{\partial}{\partial t}\delta(t; \boldsymbol{\lambda})(v_B - P_t) + \delta(t; \boldsymbol{\lambda})\frac{d}{dt}P_t + \delta(t; \boldsymbol{\lambda})[\lambda_t \pi_B(v_B) + \ell_0 \pi_B^N(v_B)] = 0$$

which simplifies to

$$\frac{\partial}{\partial t}\log \delta(t; \boldsymbol{\lambda})(k - P_t) + \frac{d}{dt}P_t + \lambda_t \pi_B(k) + \ell_0 \pi_B^N(k) = 0 \tag{27}$$

So consider an initial state (k_0, t_0) of the private game with continuation cutoff path $\mathbf{K}^{k_0,t_0}$, continuation prices $\{P_t\}_{t\geq 0} = \{\hat{P}^{pr}(K_t^{k_0,t_0}, t_0 + t|\boldsymbol{\lambda}^*)\}_{t\geq t_0-}$ and continuation (costly) intervention rates $\mathbf{L}_{t_0}^* = \{\lambda_{t_0+t}^*\}_{t\geq 0}$. Take a marginal type $v_B = k < k_0$ that trades at a time $t_0 + t$ in the interior of a smooth trade region $\bar{I} = [\underline{t}, \bar{t}) \subset (t_0, \infty)$ under that continuation. Then plugging in the expression for P_t in (27), using $\frac{d}{dt}P_t = \frac{\partial}{\partial k}\hat{P}^{pr}(K_t^{k_0,t_0}, t_0 + t|\boldsymbol{\lambda}^*)\dot{K}_t + \frac{\partial}{\partial t}\hat{P}^{pr}(K_t^{k_0,t_0}, t_0 + t|\boldsymbol{\lambda}^*)$, and Leibniz's rule, $-\dot{K}_t^{k_0,t_0} = \varphi^{pr}(K_t^{k_0,t_0}, t + t_0; \boldsymbol{\lambda}^*)$. This proves the rate expression for the private game. For the public game starting at state k_0, following the same argument but using $P_t := \hat{P}^{pu}(K_t^{k_0})$ and $\Lambda(K_s^{k_0})$ instead of λ_s^* in (27), one likewise obtains $\dot{K}_t^{k_0} = -\varphi^{pu}(K_t^{k_0})$.

Ranking Results

Proof of Theorem 3. It was shown in the text that, if type $k > 0$ trades smoothly in both the public and private games,

$$p^{pr}(k) > (<)\, p^{pu}(k) \quad \text{whenever DEI (DDI) holds.}$$

Suppose instead instead k trades in an atom in the public offers equilibrium. If the environment is DDI, $p^{pr}(k) < \hat{P}^{pu}(k) \leq p^{pu}(k)$, where the last inequality follows from $P^{pu} = \textbf{mon}\, \hat{P}^{pu}$. If the environment is DEI, then, using Theorem 5, there exists some $k^\dagger < k$ (the left endpoint of the jump that k is part of) such that $k^\dagger$ trades smoothly, and

$$p^{pu}(k) = p^{pu}(k^\dagger) = \hat{P}^{pu}(k^\dagger) < p^{pr}(k^\dagger). \tag{28}$$

By Theorem 5, $k^\dagger$ trades strictly later than k in the private game, since there is smooth trade at a strictly positive speed for all states on path. Rearranging the speed of trade in Theorem 5.3, one finds that private prices have a strictly negative *total* derivate with respect to time:

$$(\lambda_t^* + r)\left[K_t - \hat{P}^{pr}(K_t, t) - \frac{\lambda_t^*}{\lambda_t^* + r}\pi_B(K_t)\right] = -\frac{d}{dt}\hat{P}^{pr}(K_t, t)$$

where the left hand side is strictly positive by Assumption 3. Hence, $p^{pr}(k^\dagger) < p^{pr}(k)$, which together with (28) proves the DEI price ranking.

Next, I show the cutoff ranking for the DEI case—the DDI argument follows symmetrically. I claim that, whenever $K_t^{pu} = K_t^{pr}$ and both are dropping smoothly, K^{pu} is dropping faster. Since $K_{0^-}^{pu} = K_{0^-}^{pr} = 1$, and K^{pu} has downward jumps but K^{pr} always decreases smoothly, the result will follow. Let t^* be a time at which K^{pu} and K^{pr} meet, and let k^* denote their common value at that time. It suffices to show that[26]

$$\varphi^{pr}(k^*, t^*; \boldsymbol{\lambda}^*) < \varphi^{pu}(k^*) \tag{29}$$

To simplify that comparison, note that, since (k^*, t^*) is on path, $\lambda^*_{t^*} = \ell_0 G(\Pi_A(K^{pr}_{t^*})) = \Lambda(k^*)$; since the public offers RME has smooth trading at k^*, $\Lambda(k^*)$ is also the public costly intervention rate at $K^{pu}_{t^*} = k^*$, and $D(k^*) = \frac{\lambda^*_t}{\lambda^*_t + r}$. After plugging in the expressions for $\hat{P}^{pu}$ and $\hat{P}^{pr}$ from (17), the trading speeds become

$$\varphi^{pu}(k^*) = (\lambda^*_{t^*} + r)\left[\frac{k^* - \frac{\lambda^*_{t^*}}{\lambda^*_{t^*}+r}\left(\pi_B(k^*) + \pi_S(k^*)\right) - D'(k^*)\Pi_S(k^*)\frac{F_B(k^*)}{f_B(k^*)}}{D(k^*)\pi'_S(k^*) + D'(k^*)\pi_S(k^*) + \rho'(k^*)}\right]$$

$$\varphi^{pr}(k^*, t^*|\boldsymbol{\lambda}^*) = (\lambda^*_t + r)\left[\frac{k - \frac{\lambda^*_t}{\lambda^*_t+r}\left(\pi_B(k) + \pi_S(k)\right)}{D^X(t|\boldsymbol{\lambda}^*)\pi'_S(k)}\right]$$

where I have simplified using Leibniz's rule: $\frac{\partial}{\partial t}D^X(t|\boldsymbol{\lambda}^*) = -\lambda^*_t + (r + \lambda^*_t)D^X(t|\boldsymbol{\lambda}^*)$.

Consider the fractions in square brackets. The numerator on $\varphi^{pu}(k^*)$ is strictly larger than the one in $\varphi^{pr}(k^*, t^*|\boldsymbol{\lambda}^*)$, since $D'(k^*) < 0$. Log-concavity of F_B and the diminishing effects condition (Definition 9) ensures that $\rho'(\cdot) < 0$. Together with $D(k^*) < D^X(t^*|\lambda^*)$, $\pi'_S(\cdot) > 0$, and $D'(\cdot) < 0$, one has that the denominator in $\varphi^{pu}(k^*)$ must also be strictly *smaller* than the one in $\varphi^{pr}(k^*, t^*|\boldsymbol{\lambda}^*)$, which concludes the proof of the DEI case. For the DDI case, since $D'(\cdot) > 0$ and $\rho'(\cdot) > 0$, it follows that $(\hat{P}^{pu})'(k) > 0$ for all k, so the state does not jump in the public game. All the arguments above now apply symmetrically to rank $\varphi^{pu}(k^*)$ and $\varphi^{pr}(k^*, t^*|\boldsymbol{\lambda}^*)$, since log-concavity of F_B and the diminishing effects condition now ensure $\rho'(\cdot) > 0$. $\square$

Existence

I start with two workhorse lemmas that guarantee that the smooth trade paths defined by Theorem 5 are well defined.

Lemma 8. *Let $h : \mathbb{R}_+ \to \mathbb{R}$ be a continuous, monotone function that is absolutely continuous on $[t, a]$, for every $t \in (0, a)$. Then h is absolutely continuous on all of $[0, a]$.*

Proof. It suffices to show for h increasing. Then by Lebesgue's theorem on the differentiability of monotone functions, h has a derivative h' that exists almost everywhere, is (Lebesgue) integrable, and satisfies

$$h(t') - h(t) \geq \int_t^{t'} h'(s)ds, 0 \leq t \leq t' \tag{30}$$

By assumption, for $t > 0, t' \in [t, a]$, (30) must hold with equality.

Fixing $\varepsilon \in (0, t')$, since $h' \geq 0$ almost everywhere,

$$h(t') - h(\varepsilon) = \int_\varepsilon^{t'} h'(s)ds \leq \int_0^{t'} h'(s)ds \leq h(t') - h(t)$$

Taking limits as $\varepsilon \searrow 0$, (30) holds with equality for all t. $\square$

[26] (29) suffices if (k^*, t^*) is in the interior of a smooth trading region of K^{pu}. If (k^*, t^*) is at the start of a smooth trade region and $(\hat{P}^{pu})'(k^*) > 0$, the right-derivative of K^{pu} is still given by (19), and if $(\hat{P}^{pu})'(k^*) = 0$, K^{pu} is dropping at an infinite speed at t^* (with the law of motion of K^{pu} is given in a Carathéodory sense, as in Lemma 9.3.) so there is nothing to prove.

Lemma 9. *Let λ denote a smooth costly intervention rate path $t \mapsto \lambda_t$ taking values in $[\Lambda(1), \Lambda(0)]$, and let $[\underline{k}, \bar{k})$ be a non-empty interval on which $(\hat{P}^{pu})'(k) \geq 0$ with strict inequality for $k < \bar{k}$. Let $IVP^{pu}(k_0)$ and $IVP^{pr}(k_0, t_0)$ denote, respectively, the initial value problems*

$$\begin{aligned} \dot{K}_t &= -\varphi^{pu}(K_t), \qquad K_0 = k_0, \\ \dot{K}_t &= -\varphi^{pr}(K_t, t; \lambda), \quad K_{t_0} = k_0 \end{aligned} \tag{31}$$

with φ^{pu} and φ^{pr} given by Theorem 5. Then

1. *For any $k_0 \in (\underline{k}, \bar{k}]$, $IVP^{pu}(k_0)$ has a unique (forward) Caratheodory solution K_t^{pu}[27] on a maximal interval of existence $[0, t^+)$.*
   (a) *If $\underline{k} = \underline{v}$, $t^+ = \infty$, and for all $t' \in [0, \infty)$, $K_{t'}^{pu} > 0 = \lim_{t \to \infty} K_t^{pu}$.*
   (b) *If $k_0 < \bar{k}$, or if $k_0 = \bar{k}$ and $(\hat{P}^{pu})'(\bar{k}) > 0$, K_t^{pu} is C^1 for all $t \in [0, t^+)$. Otherwise, it is C^1 for $t \in (0, t^+)$.*
2. *For any (k_0, t_0), $k_0 > 0$, $IVP^{pr}(k_0, t_0)$ has a unique (forward) Caratheodory solution K_t^{pr} defined for all $t \geq 0$.*
   (a) *For all $t' \in [0, \infty)$,$K_{t_0+t'}^{pr} > 0 = \lim_{t \to \infty} K_{t_0+t}^{pr}$.*
   (b) *If $k_0 < 1$, or if $k_0 = 1, \pi_S'(1) > 0$, K_t^{pr} is C^1 for all t. Otherwise, it is C^1 for $t > 0$.*

Proof. Assumption 1 implies that φ^{pu} and φ^{pr} are C^1 in k, on $(0, k^*)$ and $(0, k')$ for any $k' < 1$, respectively. Taking derivatives on (13), using $\Pi_S(\underline{v}^+) = F_B(\underline{v}^+) = \pi_S(\underline{v}^+) = 0$, $(\hat{P}^{pu})'(\underline{v}^+) = D(\underline{v}^+)\pi_S'(\underline{v}^+) > 0$. The denominators of $\hat{\varphi}^{pu}$ and $\hat{\varphi}^{pr}$ remain strictly positive as $k \downarrow \underline{v}$, so by the quotient rule, $\hat{\varphi}^{pu}$ is C^1 in k on $[\underline{v}, k^*)$, and $\hat{\varphi}^{pr}$ is C^1 in k on $[\underline{v}, 1]$. Second, since $t \mapsto \lambda_t$ is continous and bounded away from zero, $D^X(\cdot|\lambda)$ is strictly positive, bounded, and C^1. Since $t \mapsto \lambda_t$ in fact, C^1, φ^{pr} is C^1 with respect to t for all $t \geq 0$, for $k > 0$. Therefore φ^{pu} and φ^{pr} both satisfy a local Lipschitz condition in the cutoff variable; and by the Picard-Lindelöf Theorem, $IVP^{pu}(k_0)$ and $IVP^{pr}(k_0, t_0)$ have locally unique solutions for $k_0 \in (\underline{k}, \bar{k})$ and $k_0 < 1, t_0 \geq 0$, respectively.

I prove existence of Caratheodory solutions on a maximal interval for $IVP^{pu}(k_0)$. The claims about $IVP^{pr}(k_0, t_0)$ follow by almost identical arguments and are omitted. Let $\hat{K}_t$ denote the locally unique solution to $IVP^{pu}(k_0)$. Since $\hat{\varphi}^{pu} \geq 0$, Lemma 7.2 in Teschl (2012) implies that there exist $t^+ > 0, t^- < 0$ such that (i) $\lim_{t \nearrow t^+} \hat{K}_t = 0$; and (ii) $t^- > -\infty$ and $\lim_{t \searrow t^-} \hat{K}_t = \bar{k}$, i.e. exits $[0, \bar{k})$ through $\bar{k}$ in finite time. If $\underline{k} = \underline{v}$, then $t^+ = \infty$, as 0 then becomes a critical point of the planar system.

This proves item 1 for all $k_0 < \bar{k}$, and for the $k_0 = \bar{k}, (\hat{P}^{pu})'(\bar{k}-) > 0$ case. For $k_0 = \bar{k}, (\hat{P}^{pu})'(\bar{k}-) = 0$, take the solution $\hat{K}_t$ found above, which reaches $\bar{k}$ at $t^- < 0$, and define $\tilde{K}_t := \hat{K}_{t+t^-}, t \in [0, t^+ - t^+)$. $\tilde{K}_t$ is continuous, decreasing, and absolutely continuous on $[\bar{t}, t^+)$. By Lemma 8, $\tilde{K}_t$ is absolutely continuous on $[0, t^+)$, so it is a Caratheodory solution to $IVP(\bar{k})$, as required. □

Proof of Theorem 5. Optimality for the seller is a special case of the verification arguments for the gap case (see Proposition 2 and the discussion following it).

Optimality for the audience follows immediately from $\mathcal{L}(k) = \Lambda(k)$. Optimality for the buyer follows by a direct mechanism representation of continuation play and the single-crossing properties of

[27] That is, there exists an absolutely continuous path K that satisfies $K_0 = k_0$ and $\dot{K}_t = -\hat{\varphi}^{pu}(K_t)$ almost everywhere. This is to handle the possibility that $(\hat{P}^{pu})'(\bar{k}) = 0$.

the buyer's payoffs. I show the argument for the public game—the private game is analyzed in the exact same way, expanding the notation so that the continuation payoff to depend on calendar time. Let $\mathbf{K}$ denote any continuation cutoff path of the public game, and denote $\mathbf{K}^{-1}(k) = \inf\{t \geq 0 : K_t \leq k\}$. I show that the buyer has no profitable deviations on-path assuming $\mathbf{K}$ is the continuation path starting from state $k_0 = 1$, but the same argument applies for any k_0. Define

$$\mathcal{V}(\tau, v_B) := \mathcal{U}_B^{v_B}(\tau, \mathbf{P}^{pu}(\mathbf{K}), \mathbf{L}(\mathbf{K}))$$
$$\Phi(v, v') := \mathcal{V}(K^{-1}(v'), v),$$

Φ is the payoff to v from mimicking v' given a continuation path $\mathbf{K}$. From (27), the function $\mathcal{V}$ satisfies the smooth single-crossing differences property in $(-\tau, v)$ for $\tau > 0$, and single-crossing differences everywhere. Moreover, for $\tau > 0$, it is partially differentiable in both arguments. A close look at the proof of Theorem 4.2 in Milgrom (2004) shows that this suffices for the Theorem to apply if $\mathbf{K}^{-1}(v)$ is absolutely continuous.[28] Note (i) $K^{-1}(v) = 0$ for $v \in [k^\dagger, 1]$ (and therefore is non-increasing and absolutely continuous there), and (ii) K is continuously differentiable and strictly decreasing on $(0, \infty)$, so $K^{-1}(\cdot)$ is differentiable on $[0, k^\dagger)$ with a strictly negative derivative (and also absolutely continuous on that range). By Lemma 8, K^{-1} is therefore absolutely continuous on $[0, 1]$.

By Theorem 4.2 in Milgrom (2004), if I can show

$$\Phi(v, v) = \Phi(0, 0) + \int_0^s \Phi_v(s, s)\, ds, \tag{32}$$

then incentive compatibility for $K^{-1}(\cdot)$ follows, and buyer's have no deviations on path. Using (19) and (27),

$$\dot{K}_t = -\varphi^{pu}(K_t), t > \boldsymbol{K}^{-1}(k^\dagger) \Leftrightarrow \Phi_{v'}(v, v) = 0, v \in [0, k^\dagger),$$

so $\frac{d}{dv}\Phi(v, v) = \Phi_v(v, v)$ on $[0, k^\dagger)$ on $[0, k^\dagger)$. $\Phi(v, v') = v - \hat{P}^{pu}(k^\dagger)$ for $v' \in [k^\dagger, 1]$, so $\Phi_{v'}(v, v) = 0$ and $\frac{d}{dv}\Phi(v, v) = \Phi_v(v, v) = 1$ on that range. Since $v \mapsto \Phi(v, v)$ is absolutely continuous (since K^{-1} is, and $\mathcal{V}$ is in each argument), one can therefore integrate up to obtain (32) on $[0, 1]$ (with $\Phi(0, 0) = 0$)). □

Private Offers: Existence and Uniqueness of Intervention Rates λ^*

Verification for all three players follows analogously to the public game. What remains to be shown is the existence and uniqueness of the requisite fixed point in on-path cutoff paths and audience beliefs. These follow from the next lemma:

Lemma 10. *Fix $\breve{k} \in (0, 1]$, $\breve{z} \in (D(\breve{k}), D(0))$, and consider the autonomous system*

$$\begin{aligned} \dot{k} &= \chi(k, z) := \frac{-(r + \Lambda(k))k + \Lambda(k)\left(\pi_S(k) + \pi_B(k)\right)}{z\pi_S'(k)} \\ \dot{z} &= \theta(k, z) := -\Lambda(k) + (\Lambda(k) + r)z \end{aligned} \tag{33}$$

on the domain

$$\mathcal{D} = \{(k, z) : k \in (0, \breve{k}], D(k) < z < D(0)\},$$

[28] Differentiability properties of g in his notation are only used to derive the chain rule for $\frac{d}{ds}g(\bar{x}(s), s)$ and in the definition of the smooth single-crossing differences property (which can be stated without assuming *continuous* differentiability). Moreover, it suffices in his proof that the conditions hold almost everywhere at the points of differentiability of the absolutely continuous part of the allocation $\bar{x}$.

and the associated initial value problem

$$(33) \text{ subject to } k_0 = \breve{k}, z_0 = \breve{z}. \tag{34}$$

There exists a unique value of $\breve{z}$, denoted z^*, for which (34) *has a (forward) Caratheodory solution that*

1. *exists in $\mathcal{D}$ for all $t \in \mathbb{R}_+$, and*
2. *satisfies* $\lim_{t\to\infty} e^{-rt-\int_0^t \Lambda(k_s)ds} z_t = 0$.

I explain how the existence of a fixed point of beliefs in Theorem 5 follows from the lemma before proceeding with its proof. Recall that the correctness of entrant beliefs requires that the equilibrium cutoff path, call it $\mathbf{K}^*$, and the equilibrium costly intervention rate $\boldsymbol{\lambda}^*$ satisfy, for all $t \geq 0^-$,

$$\begin{aligned} \dot{K}_t^* &= \frac{-(r+\Lambda(K_t^*))K_t^* + \Lambda(K_t^*)\left(\pi_S(K_t^*) + \pi_B(K_t^*)\right)}{D^X(t|\boldsymbol{\lambda}^*)\pi_S'(K_t^*)}, \\ \lambda_t^* &= \Lambda(K_t^*), \\ K_0 &= 1. \end{aligned} \tag{35}$$

To connect (33) and (34) to (35), fix a continuous, bounded costly intervention rate path $\lambda_t, t \geq 0$, and consider the initial value problem

$$\dot{z}_t = -\lambda_t + (\lambda_t + r)z_t, \quad z_0 = z$$

which has a global solution (e.g., Corollary 2.6 in Teschl (2012)). Letting $y_t = e^{-\int_0^t (r+\lambda_s)ds} z_t$ and multiplying on both sides by $e^{-\int_0^t (r+\lambda_s)ds} z_t$, one obtains

$$\dot{y}_t = -\lambda_t e^{-\int_0^t (r+\lambda_s)ds} z_t.$$

Integrating forwards from time t and backwards from time T,

$$y_t = \int_t^T \lambda_s e^{-\int_0^s (r+\lambda_l)dl} ds + y_T \Leftrightarrow z_t - e^{-\int_t^T (r+\lambda_s)ds} z_T = \int_t^T \lambda_s e^{-\int_t^s (r+\lambda_l)dl} ds.$$

As $T \to \infty$, the expression on the far right hand side becomes the expectation of the bounded function e^{-rt}, so it must converge. Hence, from the last display, one gleans two facts. First, $\lim_{T\to\infty} e^{-\int_0^T (r+\lambda_s)ds} z_T$ exists, and equals 0 for a unique value of z_0, which is $D^X(0|\lambda)$. Second,

$$z_t = D^X(t|\lambda)\,\forall t \quad \Leftrightarrow \quad \begin{pmatrix} \dot{z}_t = -\lambda_t + (\lambda_t + r)z_t \\ \lim\limits_{t\to\infty} e^{-\int_0^t (r+\lambda_s)ds} z_t = 0 \end{pmatrix} \quad \Leftrightarrow \quad \begin{pmatrix} \dot{z}_t = -\lambda_t + (\lambda_t + r)z_t \\ z_0 = D^X(0|\lambda) \end{pmatrix}. \tag{36}$$

Lemma 10 requires two preliminary results.

Lemma 11. *Fix $\breve{k} < 1$, and let*

$$\begin{aligned} A &:= \{(k,z) : k \in (0,\breve{k}], z = D(k)\} \\ B &:= \{(k,z) : k \in (0,\breve{k}], z = D(0)\} \\ E &:= \{(k,z) : k = \breve{k}, z \in (D(\breve{k}), D(0))\} \end{aligned}$$

denote, respectively, the lower, upper and right boundaries of $\mathcal{D}$ in (k,z)-space. Then

1. *For any $(k_0, z_0) \in A \cup B$, there exists a unique backward solution $\varphi_{k_0,z_0}(t)$ to* (33) *that exits $\mathcal{D}$ through E at a finite time $T^- \in (-\infty, 0)$.*

2. *For any $(k_0 = \breve{k}, z_0) \in E$, there exists a unique forward solution $\varphi_{\breve{k},z_0}(t)$ to* (33) *that either (i) exits $\mathcal{D}$ through $A \cup B$ in a finite time $T^+ \in (0, \infty)$, or (ii) exists in $\mathcal{D}$ for all $t \in \mathbb{R}_+$ and satisfies $\lim_{t \nearrow \infty} \varphi_{\breve{k},z_0}(t) = (0, D(0))$.*

Lemma 12. *For $\breve{k} < 1$, let E_A denote the set of $z \in (D(\breve{k}), D(0))$ for which there exists $(k_0, z_0) \in A$ such that a negative orbit of* (33) *starting at (k_0, z_0) exits $\mathcal{D}$ through $(\breve{k}, z) \in E$, and likewise for E_B. Then E_A and E_B (i) are non-empty disjoint intervals, (ii) $\sup E_A \leq \inf E_B$, (iii) $\inf E_A = D(\breve{k})$, and (iv) $\sup E_B = D(0)$.*

Proof of Lemma 11. I prove item 1; the proof of item 2 is nearly identical and left to the reader. Existence is standard: since the right hand side of (33) is C^1 (thus locally Lipschitz) on $\mathcal{D}$, for any $(k_0, z_0) \in A \cup B$, there exists a unique backward solution $\varphi_{k_0,z_0}(t) = (k_t, z_t)$ to (33) on a maximal interval of existence $(T^-, 0]$ (Theorem 2.13 in Teschl (2012)).

On int $\mathcal{D}$, $\chi < 0$ and $\theta > 0$, so $\varphi_{k_0,z_0}(t)$ only moves south-east in the (k, z) plane as $t \searrow T^-$. Since the only possible critical point of (33) in cl $\mathcal{D}$ is at $(0, D(0))$, it follows from Lemma 7.2 in Teschl (2012) that φ_{k_0,z_0} must exit $\mathcal{D}$ in finite time ($T^- \in (-\infty, 0)$), and it cannot exit through B. If φ_{k_0,z_0} were to exit through A at some point $(k, D(k))$, the orbit it traces would have to cross $D(\cdot)$ from above. However, since $D'(k) < 0$ and $\theta(k, D(k)) = 0$, the orbit is never steeper than $D(\cdot)$ at a point of intersection with A. $\square$

Proof of Lemma 12. (i) Non-emptiness follows from Lemma 11.1. A and B are disjoint, so $z \in E_A \cap E_B$ would violate the local uniqueness of forward solutions to (33). To prove connectedness, take $z', z'' \in E_A$ with $z' < z''$, and suppose some $\breve{z} \in (z', z'')$ is not in E_A. Then the forward orbit of (33) starting at $(\breve{k}, \breve{z})$ exits $\mathcal{D}$ through $B \cup \{(0, D(0)\}$, but this would mean that it intersects the forward orbit starting at $(\breve{k}, z'')$, a contradiction. E_A is therefore connected; the connectedness of E_B follows by identical arguments. (ii) If there were some $z' \in E_A$, $z'' \in E_B$ with $z' > z''$, by Lemma 11.1 we would find backward orbits of (33) that intersect inside $\mathcal{D}$. Hence $\sup E_A \leq \inf E_B$. (iii) Finally, suppose, toward a contradiction, that $\inf E_A > D(\breve{k})$, and take some $z' \in (D(\breve{k}), \inf E_A)$. Since $z' \notin E_A$, the forward orbit to (33) starting at $(\breve{k}, z')$ would need to exit $\mathcal{D}$ through B or $(0, D(0))$, but this would imply that it intersects forward orbits starting that start at $(\breve{k}, z)$, $z \in E_A$, a contradiction. A symmetric argument rules out $\sup E_B < D(0)$. $\square$

Proof of Lemma 10. Notice that the boundary condition in the second item follows immediately from global existence inside $\mathcal{D}$: if the solution stays in $\mathcal{D}$, $z_t \in (D(1), D(0)), \forall t$. Hence, by Lemma 11, it suffices to show that, for any $\breve{k} \in (0, 1]$, (i) there exists some z^* such that a forward solution to (33) starting at $k_0 = \breve{k}, z_0 = z^*$ exits $\mathcal{D}$ through $(0, D(0))$, (ii) z^* is unique, and (iii) that solution is absolutely continuous even when $\breve{k} = 1, \pi'_S(1-) = 0$.

Existence, $\breve{k} < 1$: . I show $z^* := \sup E_A$ has the required properties. Again using Lemmas 11.2 and 12, for each $z \in (D(\breve{k}), \sup E_A)$, there exists a unique $\iota_A(z) \in (D(\breve{k}), D(0))$ such that the forward solution to (33) with initial condition $k_0 = \breve{k}, z_0 = z$ exits $\mathcal{D}$ through $(D^{-1}(\iota_A(z)), \iota_A(z))$. The mapping ι_A must be weakly increasing, since distinct orbits of (33) cannot cross inside $\mathcal{D}$; it must be strictly increasing, since $z' < z''$ with $\iota_A(z') = \iota_A(z'')$ would violate uniqueness of backward solutions to (33) starting at $k_0 = D^{-1}(\iota_A(z')), z_0 = \iota_A(z')$. Moreover, $\bar{\iota} := \lim_{z \nearrow z} \iota_A(z) = D(0)$. $\bar{\iota} \leq D(0)$ is immediate from the definition of $\mathcal{D}$. If $\bar{\iota} < D(0)$, then for small enough $\varepsilon > 0$, the point $y = (D^{-1}(\bar{\iota}+\varepsilon), \bar{\iota}+\varepsilon)$ is in A. Since the orbits of (33) cannot intersect, the backward orbit of (33) starting at y would exit $\mathcal{D}$ through E at a point strictly above $(\breve{k}, \sup E_A)$, which contradicts the definition of $\sup E_A$.

So let $(\tilde{k}, \tilde{z})$ denote the point at which the forward solution to (33) starting at $(\breve{k}, \sup E_A)$ reaches bd $\mathcal{D}$ (either asymptotically or in finite time). That point is unique (by local uniqueness of solutions to (33)). Since orbits of (33) cannot intersect inside $\mathcal{D}$, $(\tilde{k}, \tilde{z})$ must satisfy, for any $\varepsilon > 0$,

$$\iota_A(z-\varepsilon) \le \tilde{z} \le D(0),$$
$$0 \le \tilde{k} \le D^{-1}(\iota_A(z-\varepsilon)).$$

Taking limits as $\varepsilon \searrow 0$, we obtain, using $D'(\cdot) < 0$, $(\tilde{k}, \tilde{z}) = (0, D(0))$.

Uniqueness, $\breve{k} < 1$.: Suppose there existed $\underline{z}, \bar{z} \in (D(1), D(0)), \underline{z} < \bar{z}$ such that (33) has a forward solution inside $\mathcal{D}$ with the required boundary condition for two initial conditions $k_0 = \breve{k}, z_0 = \underline{z}$ and $k_0 = \breve{k}, z_0 = \bar{z}$. Denote the respective solution paths by $(\underline{k}_t, \underline{z}_t)$ and $(\bar{k}_t, \bar{z}_t)$, and let $\underline{\lambda}_t := \Lambda(\underline{k}_t)$ and similarly for $\bar{\lambda}_t$.

Let T^+ denote the largest time for which both solutions exist. First, I claim that for any $t \le T^+$, $\bar{k}_t \ge \underline{k}_t$, strictly for $t \in (0, T^+)$, which implies that $\bar{\lambda}_t \le \underline{\lambda}_t$, strictly for $t \in (0, T^+)$. Suppose there were some $t^* \in (0, T^+)$ with $\bar{k}_{t^*} = \underline{k}_{t^*}$ (if there are many, choose the earliest such moment). Since $\bar{k}_0 = \underline{k}_0$ and $\bar{z}_0 > \underline{z}_0$, the expression for θ in (33) implies that $\lim_{t \searrow 0} \dot{\bar{k}}_t < \lim_{t \searrow 0} \dot{\underline{k}}_t$, so at t^*, $\bar{k}_t$ must be crossing $\underline{k}_t$ from above, i.e. $\dot{\underline{k}}_{t^*} \le \dot{\bar{k}}^*_t$. $\dot{\underline{k}}_{t^*} = \dot{\bar{k}}^*_t$ would violate uniqueness of *backwards* solutions to (33), so from the expression for χ in (33), $\bar{k}_{t^*} = \underline{k}_{t^*}$ requires $\bar{z}_{t^*} < \underline{z}_{t^*}$. The latter is impossible, since it would require the orbit traced by $(\bar{k}_t, \bar{z}_t)$ to have crossed the orbit traced by $(\underline{k}_t, \underline{z}_t)$ inside $\mathcal{D}$.

Second, from (36) and the related discussion, the system $\dot{z}_t = -\underline{\lambda}_t + (\underline{\lambda}_t) z_t, z_0 = \bar{z}$ has a solution $\tilde{z}_t$ for which $\lim_{t\to\infty} e^{-\int_0^t (r+\underline{\lambda}_s)ds} \tilde{z}_t > 0$. I claim that $\tilde{z}_t \le \bar{z}_t$ for all $t \ge 0$. Using $\underline{\lambda}_t > \bar{\lambda}_t, t > 0$, one obtains that, at any point $t' > 0$ at which $\tilde{z}_{t'} = \bar{z}_{t'}$,

$$\dot{\tilde{z}}_{t'} = r\tilde{z}_{t'} + \underline{\lambda}_{t'}(\tilde{z}_{t'} - 1) < r\bar{z}_{t'} + \bar{\lambda}_t(\bar{z}_{t'} - 1) = \dot{\bar{z}}_{t'}$$

where the inequality uses the fact that $\bar{z}_t = \int_t^\infty \bar{\lambda}_s e^{-\int_t^s (r+\bar{\lambda}_l)dl} ds \in (0,1)$. Therefore $\tilde{z}_t$ only crosses $\bar{z}_t$ from above; since $\bar{z}_0 = \tilde{z}_0$, the ranking follows.

The above implies that, for all $t > 0$,

$$e^{-\int_0^t (r+\bar{\lambda}_s)ds} \bar{z}_t > e^{-\int_0^t (r+\underline{\lambda}_s)ds} \tilde{z}_t > 0$$

and thus one obtains the contradiction

$$\lim_{t\to\infty} e^{-\int_0^t (r+\bar{\lambda}_s)ds} \bar{z}_t > \lim_{t\to\infty} e^{-\int_0^t (r+\underline{\lambda}_s)ds} \tilde{z}_t > 0.$$

$\breve{k} = 1, \pi'_S(1-) = 0$ case: If $\pi'_S(1-) > 0$, the argument above follows identically for $\breve{k} = 1$. It remains to show existence, uniqueness and absolute continuity of the solution hold even if $\breve{k} = 1, \pi'_S(1-) = 0$. The argument starts with $\breve{k} < 1$ and finds the right kind of limit.

So consider any $\breve{k} < 1$, and let $z^*_{\breve{k}}, \mathcal{D}_{\breve{k}}$ and $E_{\breve{k}}$ be as in Lemmas 10 and 11, with the dependence on $\breve{k}$ made explicit. By identical arguments to those given in Lemma 11, starting from $k_0 = \breve{k}$ and $z_0 = z^*_{\breve{k}}$, (33) has a unique solution $(k(t), z(t))$ in $\mathcal{D}_1$ on a maximal interval of existence (T^-, T^+). Moreover, by Lemma 11.1, $T^- \in (-\infty, 0)$, $\lim_{t\searrow T^-} k(t) = 1$ and $\lim_{t\searrow T^-} z(t) \in (D(1), D(0))$. Define $z^*_1 := \lim_{t\searrow T^-} z(t)$. The functions

$$
\begin{aligned}
k^\circ(t) &:= k(t+T^-), k^\circ(0) := 1, \\
z^\circ(t) &:= z(t+T^-), z^\circ(0) := z_1^*,
\end{aligned}
\tag{37}
$$

defined for $t \geq 0$, are therefore continuous forward trajectories that start in E^1 and exit $\mathcal{D}^1$ through $(0, D(0))$. This establishes existence for $\breve{k} = 1$. If a forward trajectory $k_0 = 1, z_0 \in (D(1), D(0))$ with $z_0 \neq z_1^*$ existed globally and reached $(0, D(0))$, it would have to intersect the orbit traced by $k^\circ(t), z^\circ(t)$ at some $t > 0$, a contradiction. Thus the trajectory has a unique starting point in E_1.

Next, notice that $(k^\circ(t), z^\circ(t))$ are continuous, monotone and C^1 for $t > 0$. Moreover, for $t > 0, t' > t$, (k°, z°) solve (33) classically. Hence, since χ is continuous for $k \in [0,1), z \in (D(1), D(0))$, k° satisfies the integral equation

$$
k^\circ(t') - k^\circ(t) = \int_t^{t'} \chi(k(t+T^-), z(s+T^-))ds, \; 0 < t \leq t'.
$$

Absolute continuity on $[0, \infty)$ then follows from Lemma 8. An identical procedure yields the absolute continuity of z°. $\square$

B Gap Case, Removing Assumption 4.2

Here I describe results for the gap case. This section covers definitions, necessary conditions and ranking results comparing the public and private games, and equilibrium constructions for the public game, all under Assumption 4.1 Constructions of off-path behavior for the private game are in Online Appendix A.2 The public no-gap case without Assumption 4.1 is treated in Online Appendix A.4; the case with a gap and removing Assumption 4.1 is straightforward from there and omitted, to avoid redundancy.

The construction for both the gap case and the case of general $\hat{P}^{pu}$ borrows heavily from the continuous time "Weak Markov" equilibrium proposed in Chaves and Varas (2022).[29] Incorporating the gap case requires an expanded strategy for the seller and an expanded equilibrium notion.

Definition 10 (Seller Strategy, Gap)**.** A plan for the seller is a cutoff path $\mathbf{K}$ and a stopping time T at which to make the pooling offer $P^{pu}(\underline{v})$. The seller can mix over stopping times T according to a CDF $H = (H_t)_{t \geq 0}$. Say that the stopping time mix H is *Markov* if its hazard measure is a function of K_{t-} only. A plan $(\mathbf{K}, H)$ is admissible if $\mathbf{K}$ is admissible according to definition 4, and $\mathcal{A}_k^{ga}$ denotes the set of admissible plans with cutoff paths starting at k, i.e., $K_{0^-} = k$.

For a seller plan $(\mathbf{K}, H)$, say there is *smooth trade* a *smooth trade region* is a time interval $[\underline{t}, \bar{t})$ where $t \mapsto K_t$ is absolutely continuous and H is constant, i.e., trade is gradual and the pooling offer is not used. For any such region, $\dot{K}_t$ denotes the (a.e.) derivative of K_t.

Given a cutoff path $\mathbf{Q}$ and a realization of the stopping time T, the seller's expected payoffs are, for the interruption case,

$$
\begin{aligned}
\mathcal{J}(\mathbf{Q}, T) := & \mathbb{E}_k^{\mathbf{Q}} \Bigg[\int_0^{\sigma \wedge T} e^{-rt} P^{pu}(Q_t) d(1 - F_B(Q_t; k)) \\
& \qquad + e^{-r\sigma} \mathbf{1}_{\sigma \leq T} F_B(Q_\sigma; k) \Pi_S(Q_\sigma) + e^{-rT} \mathbf{1}_{\sigma > T} F_B(Q_T; k) P^{pu}(\underline{v}) \Bigg].
\end{aligned}
\tag{38}
$$

[29] See that paper for additional details. I thank Felipe Varas for many insightful conversations on the topic.

The seller therefore solves

$$\sup_{(\mathbf{Q},H)\in\mathcal{A}_k^U}\int_0^\infty \mathcal{J}(\mathbf{Q},T)dH(T). \tag{39}$$

Buyer v_B's payoffs from stopping at τ for a path K and stopping time T are

$$\mathbb{E}_k^K[e^{-r\tau}\mathbf{1}_{\tau\leq T\wedge\sigma}(v_B-P^{pu}(K_\tau^k)+e^{-r\sigma}\mathbf{1}_{\sigma<\tau\wedge T}\pi_B(v_B)+e^{-rT}\mathbf{1}_{T<\tau\wedge\sigma}(v_B-P^{pu}(\underline{v})]$$

Let $B(v_B,k)$ denote the value function for buyer v_B at state k.

Definition 11 (Weak Markov Equilibrium, Public Gap)**.** A Weak Markov Equilibrium is an on-path tuple

$$(\{K^k\}_{k\in[\underline{v},1]},H,P^{pu},\mathcal{L}),$$

and an off-path offer mix ς for the seller, and value functions $J^{pu}(\cdot)$ for the seller and $B(\cdot,\cdot)$ for the buyer, such that

1. (K^k,H) solve (39) for each k and deliver $J^{pu}(k)$.
2. For any k, stopping at $\inf\{t:P^{pu}(K_t^k)\leq P^{pu}(v_B)\}$ is optimal for v_B, for all $v_B\in[\underline{v},1]$ and delivers value $B(v_B,k)$.
3. $\mathcal{L}(k)=\Lambda(k)$.
4. $P^{pu}(\underline{v})=\underline{v}-D(\underline{v})\pi_B(\underline{v})$.
5. For any point of discontinuity of P^{pu}, k', and any off-equilibrium offer $p'\in(P^{pu}(k'+),P^{pu}(k'-))$, $\varsigma(\cdot|k',p')$ maximizes[30]

$$\int_0^\infty \tilde{p}\left\{(1-F_B((P^{-1}(\tilde{p});k')))^+ \right.$$
$$\left. +F_B((P^{pu})^{-1}(\tilde{p});k')J^{pu}\left(P^{-1}(\tilde{p})\right)\right\}d\varsigma(\tilde{p}|k',p')$$

and satisfies

$$k' - p' \leq B(k',k')\int_{P^{pu}(k'-)}^\infty d\varsigma(\tilde{p}|k',p') + \int_0^{P^{pu}(k'-)}(k'-\tilde{p})\,d\varsigma(\tilde{p}|k',p')$$

Condition 5, borrowed from Chaves and Varas (2022), models off-path behavior (i.e., offers that are not called for at any (k,t) by "stopping the clock": after a "mistaken" offer p', the seller can mix over future offers according to a lottery ς, but that mix must rationalize the buyer's choice to accept according to P^{pu} on path. The lottery can depend on both the state *and* the previous mistaken price; hence the equilibrium is "Weak Markov" in the sense of Fudenberg et al. (1985) and Gul et al. (1986).

Remark 4 (Private Offers Weak Markov Equilibrium)**.** The equilibrium notion for the private offers gap case is obtained from the public game with the usual changes (e.g., prices P^{pr} are a function of both (k,t) and the seller takes the costly intervention rate as exogenous; the formal definitions are evident from the no-gap case). Therefore a tuple $\left(\{K\}_{k\in[\underline{v},1],t\in[0^-,\infty)}^{k,t},H,P^{pr},\{\lambda_t^*\}_{t\geq0}\right)$ together with buyer and seller value functions and an off-path seller lottery ς are a private Weak Markov equilibrium if they satisfy the optimality conditions from Definition 11 (suitably modified for the two-dimensional state (k,t)) and

- $P^{pr}(\underline{v},t)=\underline{v}-D^X(t|\boldsymbol{\lambda}^*)\pi_B(\underline{v})$

[30]Here, $(P^{pu})^{(-1}(\cdot)$ represents the generalized inverse defined as $(P^{pu})^{-1}(y)\equiv\sup\{x>0:P^{pu}(x)\geq y\}$.

- $\lambda_t^* = \Lambda(K^{k=1,t=0^-})$.

The definition of *regularity* changes slightly. Once again it is the restriction is designed to allow for necessary conditions to be derived from standard analysis of the HJB's:

Definition 12. A Weak Markov equilibrium in the public game is regular if

1. J^{pu} is continuous everywhere and C^1 at k's for which the continuation strategy prescribes smooth trade.
2. At states with smooth trade, P^{pu} is continuous.
3. Jumps in the cutoffs are isolated, and cutoff paths are differentiable in the interior of smooth trade regions.

For private offers, regularity implies 1-3 above (for J^{pr} and P^{pr}) with the following modifications:

- J^{pr} is C^1 at (k,t)'s with smooth trade.
- For any on-path state (k,t) of the private game, the seller's continuation strategy has *absorbing smooth trade*: if trade is smooth at (k,t), it is smooth at $(k,t'), t' \geq t$.

As above, I use the shorthand "equilibrium" for "Weak Markov Regular equilibrium."

I focus on the more interesting case where the public game continues to have non-trivial dynamics:

Assumption 5 (Non-trivial public dynamics)**.** Let $k^\dagger$ be as in Assumption 4.1 and $k^{pu} := \inf\{k \geq \underline{v} : R^{pu}(k) = P^{pu}(\underline{v})\}$. Then $k^\dagger > k^{pu}$.

As shown below, the public game with a gap ends instantly at states $k < k^{pu}$, whereas $k^\dagger$ is the state the public equilibrium jumps to (this is established for the gap case below). Then, intuitively, if $k^\dagger < k^{pu}$ the public game cannot have any gradual trade.

Under this assumption, most of the necessary conditions from the no-gap analysis extend, and most of the key rankings of payoffs, prices, and trading speeds continue to hold.

Theorem 6. *Assume there is a gap, but Assumptions 1-3, 4.1 still hold.*

1. *Theorem 1 (payoff ranking at on-path smooth trade states) continues to hold for all sufficiently high k's, and holds with weak inequalities for all k's. In particular, if the public game starts with smooth trade, the seller prefers private (public) bargaining under DEI (DDI).*
2. *If, in addition, Assumption 5 holds, then*
   (a) *Theorem 2 continues to hold.*
   (b) *There exists $T^* > 0$ and a positive measure of cutoffs that are reached before T^* in either game, for which the results in Theorem 3 and Corollary 1 (1 and 2(c)) continue to hold.*

Lemma 13. *Consider the smooth trading payoff expressions in Lemma 1. With a gap, the expression for J^{pu} remains true; the expression for J^{pr} remains true for (k,t)'s with smooth trade that are reached on path.*[31]

Proof. Consider the (necessary) HJB in a private game of interruption (the remaining cases are identical). In the interior of smooth trade regions,

[31] Recall these are the states at which regularity of equilibrium has bite.

$$rJ_S^{pr}(k,t) = \sup_{(|\dot{k}|,\gamma)\in\mathbb{R}^2_+} \Bigg\{ \left(P^{pr}(k,t) - J_S^{pr}(k,t)\right) \frac{f_B(k)}{F_B(k)}|\dot{k}| + \gamma\left(P^{pr}(\underline{v},t) - J_S^{pr}(k,t)\right)$$
$$+ \lambda_t^*[\Pi_S(k) - J_S^{pr}(k,t)] + \frac{\partial}{\partial k}J_S^{pr}(k,t)(-|\dot{k}|) + \frac{\partial}{\partial t}J_S^{pr}(k,t)\Bigg\}$$
$$= \lambda_t^*[\Pi_S(k) - J_S^{pr}(k,t)] + \frac{\partial}{\partial t}J_S^{pr}(k,t) \quad (40)$$

where γ is the rate at which the pooling offer is made. The last equality follows from the linearity and the fact that $|\dot{k}| < +\infty, \gamma = 0$ must be optimal by hypothesis. Since the equilibrium is regular, the ODE above holds for all $t' > t$. Integrating backward and taking limits (using the fact that J^{pr} is bounded) one obtains $J^{pr}(k,t) = R^{pr}(k,t|\lambda^*)$. By identical arguments, $J^{pu}(k) = R^{pu}(k)$ at states with smooth trade. □

Next, I show that one can split the state space and apply the necessary conditions from the no-gap case to high enough states:

Lemma 14. *Fix regular equilibria for the public and private games. Let k^{pu} be as in Assumption 5, and let $\tilde{k}(t)$ be the solution of*

$$R^{pr}(\tilde{k}(t),t|\lambda^*) = D^X(t|\boldsymbol{\lambda^*})\Pi_S(\tilde{k}(t)) = \underline{v} - D^X(t|\boldsymbol{\lambda^*})\pi_B(\underline{v}) = P^{pr}(\underline{v},t) \quad (41)$$

with $P^{pu}(\underline{v}), P^{pr}(\underline{v},t)$ as in Definition 11 and R^{pu}, R^{pr} as the smooth trade payoffs from (10).

1. *For $k > k^{pu}$, $J^{pu}(k) > P^{pu}(\underline{v})$, and making a pooling offer at k is strictly suboptimal for the seller in the public game. For $k < k^{pu}$, the game ends immediately with a pooling offer, and $J^{pu}(k) = P^{pu}(\underline{v})$.*
2. *For $k > \tilde{k}(t)$, $J^{pr}(k,t) > P^{pr}(\underline{v},t)$, and making a pooling offer at state (k,t) is strictly suboptimal for the seller in the private game.*
3. $\max\{k^{pu}, \tilde{k}(1)\} < 1$.

Proof. For the first item, if $J^{pu}(k) > P^{pu}(\underline{v})$, there is nothing to prove, so focus on $J^{pu}(k) = P^{pu}(\underline{v})$. One strategy available to the seller at k is to jump the state to $k' \in (k^{pu}, k)$ with $R^{pu}(k') > P^{pu}(\underline{v})$; that such a k' exists follows from $k > k^{pu}$. Then

$$P^{pu}(\underline{v}) = J^{pu}(k) \geq (1 - F_B(k';k))P^{pu}(k') + F_B(k';k)J^{pu}(k').$$

The seller could achieve $R^{pu}(k')$ by stalling indefinitely, so $J^{pu}(k') \geq R^{pu}(k') > P^{pu}(\underline{v})$. Hence, using the last display, $P^{pu}(k')$ must be strictly smaller than $P^{pu}(\underline{v})$, a contradiction. Next, suppose there existed some $k < k^{pu}$ for which the game did *not* end with an immediate pooling offer. By definition, $J^{pu}(k) \geq P^{pu}(\underline{v}) > R^{pu}(k)$ and $R^{pu}(k') < P^{pu}(\underline{v})$ for all $k' \leq k$. The latter, however, means that the continuation game after k cannot have smooth trade or quiet periods with positive probability, since these would give the seller less than the pooling offer $P^{pu}(\underline{v})$. The game must therefore end immediately with probability one, a contradiction.

An identical argument establishes the first statement of Point 2, replacing k^{pu} with $\tilde{k}(t)$, $R^{pu}(\cdot)$ with $R^{pr}(\cdot,\cdot|\lambda^*)$, and $P^{pu}(\underline{v})$ with $P^{pr}(\underline{v},t)$.

The third item follows directly from Assumptions 2 and 3. □

Proof of Theorem 6. Lemma 3.1 in the no-gap case (no cutoff jumps in the private offer equilibrium) only used the expression for seller payoffs. By Assumption 3, the cutoff in the private game must drop eventually (stalling at $k = 1$ forever is not an equilibrium), so under DEI (DDI) λ_t^* is strictly above (below) $\Lambda(1)$ for some large enough t. This was all that was required for Theorem 1.

By Lemma 14, since $\max\{k^{pu}, \tilde{k}(t)\} < 1$ and $k^{pu} < k^\dagger$, for high enough cutoffs, the no-gap analysis applies identically: there are no quiet periods in either game for those states, smooth-trade prices and speeds are as before, and the public game starts with a jump to $k^\dagger$ followed by smooth trade. In particular, in the public game, k^{pu} will be reached at a strictly positive time; together with no jumps in the private game, this implies the existence of some $T^* > 0$ such that, on path, the pooling offer is never used in either game until time $t = T^*$. Hence, the ranking arguments in Theorem 2, Theorem 3, and Corollary 1 apply identically for states that are reached before time T^*. □

Construction and Uniqueness for Public Gap Case

Proposition 1. *Under Assumptions 1-3, 4.1 and 5*

The following is the unique on-path Weak Markov equilibrium outcome of the public game:

1. *Prices are given by*

$$P^{pu}(k) = \begin{cases} \hat{P}^{pu}(k^\dagger), & k \in [k^\dagger, 1] \\ \hat{P}^{pu}(k), & k \in [k^{pu}, k^\dagger) \\ P^{pr}(\underline{v}) := \underline{v} - D(\underline{v})\pi_B(\underline{v}), & k < k^{pu} \end{cases}$$

2. *Cutoff policies are given by*
   - *For $k \in (k^{pu}, k^\dagger]$, trade smoothly at speed $\dot{k} = -\varphi^{pu}(k)$, where φ^{pu} is given by* (19)*, and make no pooling offers.*
   - *For $k \in (k^\dagger, 1]$, jump the state to $k^\dagger$ immediately.*
   - *For $k = k^{pu}$, keep the cutoff constant and make the pooling offer at a Poisson rate γ satisfying*

$$k^{pu} - P^{pu}(k^{pu}) = \frac{\Lambda(k)}{\gamma + r + \Lambda(k^{pu})}\pi_B(k^{pu}) + \frac{\gamma}{\gamma + r + \Lambda(k^{pu})}(k^{pu} - P^{pu}(\underline{v})) \quad (42)$$

   - *For $k < k^{pu}$, make the pooling offer immediately.*

If $k^{pu} \geq k^\dagger$, there is a Weak Markov equilibrium with prices

$$P^{pu}(k) = \begin{cases} \hat{P}^{pu}(k^{pu}), & k \geq k^{pu} \\ P^{pu}(\underline{v}) = \underline{v} - D(\underline{v})\pi_B(\underline{v}), & k < k^{pu} \end{cases}$$

and the following cutoff policy:

- *For $k > k^{pu}$, jump the state to k^{pu} immediately.*
- *At $k = k^{pu}$, keep the cutoff constant, and make the pooling offer at a Poisson rate given by* (42).
- *At $k < k^{pu}$, make the pooling offer immediately.*

The seller's payoffs before the impasse are therefore the same as in the no-gap case.

Proof of Proposition 1, Uniqueness. By Lemma 14, for $k > k^{pu}$, the seller never uses the pooling offer, and the game can be analyzed exactly as in the no-gap case. The uniqueness claims for the no-gap case therefore carry over for those states: $P^{pu}(k) = \textbf{mon}\, \hat{P}^{pu}(k), k \in (k^{pu}, 1]$ by Lemma 6, which itself pins down the cutoff paths for $k > k^{pu}$. For $k < k^{pu}$, the unique equilibrium outcome is an immediate pooling offer (using Lemma 14), while for $k = k^{pu}$, the seller is indifferent over mixing policies, but given play for $k \neq k^{pu}$, the rate in Proposition 1 is pinned down uniquely by the marginal buyer's indifference condition. $\square$

Existence, Public Gap Case

I give details for $k^{pu} < k^{\dagger}$, since the remaining case is similar.

Buyer optimality is by construction. In the pooling region, no buyer type can improve on immediate acceptance, and the mixing rate in (42) is calibrated to make $v_B = k^{pu}$ just indifferent at state k^{pu}. For the remaining types and states, the arguments in the no-gap case carry over identically.

It remains to show seller optimality. First, I rewrite (39) in a convenient form, for the case where H is absolutely continuous. [32] Let $\gamma_s := f(s)/1 - H(s)$ denote the hazard rate for such an H at s, with $\gamma := \{\gamma_t, t \geq 0\}$ and $\Gamma_t := \int_0^t \gamma_s$. Also, for an arbitrary admissible cutoff policy $\mathbf{Q} = \{Q_t, t \geq 0\}$, let $\mathbf{L}(\mathbf{Q})_t := \int_0^t \Lambda(Q_s)ds$. Then, using integration by parts, the payoff from $(\mathbf{Q}, H)$ is

$$\int_0^{\infty} e^{-rt-\mathbf{L}(\mathbf{Q})_t-\Gamma_t} P^{pu}(Q_t) d(1 - F_B(Q_t; k)) \\ + \int_0^{\infty} e^{-rt-\mathbf{L}(\mathbf{Q})_t-\Gamma_t} F_B(Q_t) \left[\gamma_t P^{pu}(\underline{v}) + \Lambda(Q_t)\Pi_S(Q_t)\right] dt \quad (43)$$

To interpret the expression, consider for example the second integral. Notice that T and σ are independent conditional on $\mathbf{Q}$, so $\mathbb{P}^{\mathbf{Q},\gamma}(\sigma \wedge T > t) = e^{-\mathbf{L}(\mathbf{Q})_t} e^{-\Gamma_t}$. Hence, the joint probability that $T \in [t, t + dt)$ and $\sigma > t + dt$ is $\approx \gamma_t e^{-\Gamma_t} dt \times e^{-\mathbf{L}(\mathbf{Q})_t}$, which is the factor multiplying $F_B(Q_t)P(\underline{v})$.

Proposition 2. *Let* $(\mathbf{Q}^*, H^*)$ *be an admissible policy for problem* (39)*, and let* $\psi : [\underline{v}, 1] \to \mathbb{R}$ *be its implied value function. If*

1. ψ *is continuous, has a continuous derivative on* $[\underline{v}, \underline{k})$ *and* $(\underline{k}, 1]$*, with a possible kink at* $\underline{k}$*;*
2. *(OPT-J) For all* $k \in [\underline{v}, 1]$*,*
$$\psi(k) \geq (1 - F_B(k'; k))P^{pu}(k') + F_B(k'; k)\psi(k') \quad \forall k' \leq k.$$
3. *(OPT-S) For almost every* $k \in [\underline{v}, \underline{k})$ *and almost every* $k \in (\underline{k}, 1]$*,*
$$u\left[\frac{f_B(k)}{F_B(k)}(P^{pu}(k) - \psi(k)) - \psi'(k)\right] + w\left[P^{pu}(\underline{v}) - \psi(k)\right] \\ - (r + \Lambda(k))\psi(k) + \Lambda(k)\Pi_S(k) \leq 0 \;\forall (u, w) \in \mathbb{R}_+^2,$$

then $(\mathbf{Q}^*, H^*)$ *solves* (39).

Proof. Suppose ψ satisfies items 1 to 3. Start with $k \in [\underline{k}, 1]$, and take an arbitrary admissible policy $(\mathbf{Q}, H)$ for (39) such that H is absolutely continuous with hazard rate path γ. Standard change-of-value formulas then yield

[32] The formulation will suffice for checking seller optimality even if H is not absolutely continuous, since the payoff of such policies can be approximated by sequences of policies with absolutely continuous H's in the weak star topology.

$$
\begin{aligned}
e^{-rt-L(\mathbf{Q})_t-\Gamma_t}F_B(Q_t;k)\psi(Q_t)-\psi(k) = \\
&\sum_{s<t} e^{-rs-\mathbf{L}(\mathbf{Q})_s-\Gamma_s}\left[F_B(Q_s;k)\psi(Q_s)-F_B(Q_{s-};k)\psi(Q_{s-})\right] \\
&+\int_0^t e^{-rs-\mathbf{L}(\mathbf{Q})_s-\Gamma_s}\left[\frac{f_B(Q_s)}{F_B(Q_s)}\psi(Q_s)\dot{Q}_s+F_B(Q_s;k)\psi'(Q_s)\dot{Q}_s\right. \\
&\qquad\left.-(-rs-\mathbf{L}(\mathbf{Q})_s-\Gamma_s)F_B(Q_S;k)\psi(Q_s)\right]dt \\
&\leq \\
&-\int_0^t e^{-rs-\mathbf{L}(\mathbf{Q})_s-\Gamma_s}P^{pu}(Q_s)d(1-F_B(Q_s;k) \\
&-\int_0^t e^{-rs-\mathbf{L}(\mathbf{Q})_s-\Gamma_s}F_B(Q_s)\left[\gamma_s P(\underline{v})+\Lambda(Q_s)\Pi_S(Q_S)\right]ds
\end{aligned}
$$

Taking limits as $t\to\infty$, we find that $\psi(k)$ is greater than the payoff from $(\mathbf{Q},H)$. Since $(\mathbf{Q},H)$ was arbitrary, $(\mathbf{Q}^*,H^*)$ weakly dominates all admissible policies with absolutely continuous H's; those policies can approximate any admissible policies arbitrarily closely, so $(\mathbf{Q}^*,H^*)$ is optimal in (39) starting from $k\in[\underline{v},\underline{k}]$.

Now take $k\in(\underline{k},1]$. Since cutoffs are always monotonically decreasing, the principle of optimality implies that one can write objective (43) as follows:

$$
\begin{aligned}
\sup_{\mathbf{Q}\in\mathcal{A}_k,\gamma}\int_0^{\tau(\mathbf{Q})} e^{-rt-\mathbf{L}(\mathbf{Q})_t-\Gamma_t}P^{pu}(Q_t)d(1-F_B(Q_t;k)) \\
+\int_0^{\tau(\mathbf{Q})} e^{-rt-\mathbf{L}(\mathbf{Q})_t-\Gamma_t}F_B(Q_t)\left[\gamma_t P(\underline{v})+\Lambda(Q_t)\Pi_S(Q_t)\right]dt \\
+e^{-r\tau(\mathbf{Q})-\mathbf{L}(\mathbf{Q})_{\tau(\mathbf{Q})}-\Gamma_{\tau(\mathbf{Q})}}\psi(Q_{\tau(\mathbf{Q})})
\end{aligned}
\tag{44}
$$

where $\tau(\mathbf{Q}):=\inf\{t>0:Q_t\leq\underline{k}\}$ is the first time the cutoff reaches $\underline{k}$. Recreating the previous steps, assume ψ satisfies (OPT-J) and (OPT-S), and take an arbitrary admissible policy $(\mathbf{Q},\gamma)$, $\mathbf{Q}\in\mathcal{A}_k$. Again, by change-of-value formulas,

$$
\begin{aligned}
e^{-rt\wedge\tau(\mathbf{Q})-L(\mathbf{Q})_{t\wedge\tau(\mathbf{Q})}-\Gamma_{t\wedge\tau(\mathbf{Q})}}F_B(Q_{t\wedge\tau(\mathbf{Q})};k)\psi(Q_{t\wedge\tau(\mathbf{Q})})-\psi(k) = \\
&\sum_{s<t\wedge\tau(\mathbf{Q})} e^{-rs-\mathbf{L}(\mathbf{Q})_s-\Gamma_s}\left[F_B(Q_s;k)\psi(Q_s)-F_B(Q_{s-};k)\psi(Q_{s-})\right] \\
&+\int_0^{t\wedge\tau(\mathbf{Q})} e^{-rs-\mathbf{L}(\mathbf{Q})_s-\Gamma_s}\left[\frac{f_B(Q_s)}{F_B(Q_s)}\psi(Q_s)\dot{Q}_s+F_B(Q_s;k)\psi'(Q_s)\dot{Q}_s\right. \\
&\qquad\left.-(-rs-\mathbf{L}(\mathbf{Q})_s-\Gamma_s)F_B(Q_S;k)\psi(Q_s)\right]dt \\
&\leq \\
&-\int_0^{t\wedge\tau(\mathbf{Q})} e^{-rs-\mathbf{L}(\mathbf{Q})_s-\Gamma_s}P^{pu}(Q_s)d(1-F_B(Q_s;k)) \\
&-\int_0^{t\wedge\tau(\mathbf{Q})} e^{-rs-\mathbf{L}(\mathbf{Q})_s-\Gamma_s}F_B(Q_s)\left[\gamma_s P(\underline{v})+\Lambda(Q_s)\Pi_S(Q_S)\right]ds
\end{aligned}
$$

Taking $t \to \infty$ on both sides, $t \wedge \tau(\mathbf{Q}) \to \tau(\mathbf{Q})$, and starting from $k \in (\underline{k}, 1]$, $(\mathbf{Q}^*, H^*)$ weakly dominates any $(\mathbf{Q}, H)$ with absolutely continuous H. By the arguments above, $(\mathbf{Q}^*, \gamma^*)$ must solve (39) starting from $k \in (\underline{k}, 1]$. $\square$

I now show that the value seller policy described in Proposition 1 is a best response to P^{pu} and $\mathcal{L} = \Lambda$. I explain the case $k^{pu} < \hat{k}$, as the remaining case is similar. That policy gives payoffs

$$\psi^*(k) = \begin{cases} L(k, \hat{k}), & k \in (\hat{k}, 1] \\ R^{pu}(k), & k \in [k^{pu}, \hat{k}] \\ P^{pu}(\underline{v}), & k \in [\underline{v}, k^{pu}). \end{cases} \tag{45}$$

ψ^* is continuous everywhere. It is smooth on $[\underline{v}, k^{pu})$, $(k^{pu}, \hat{k})$ and $(\hat{k}, 1]$. It is also C^1 at $\hat{k}$: for $k > \hat{k}$,

$$(\psi^*)'(k) = \frac{F_B(\hat{k})}{F_B(k)^2} f_B(k) \left[P^{pu}(\hat{k}) - D(\hat{k})\Pi_S(\hat{k})\right]. \tag{46}$$

By the continuity of f_B, the above implies that

$$(\psi^*)'(\hat{k}^+) = \frac{f_B(\hat{k})}{F_B(\hat{k})} \left[P^{pu}(\hat{k}) - D(\hat{k})\Pi_S(\hat{k})\right].$$

On the other hand, from (12), $(\psi^*)'(\hat{k}^-) = \frac{f_B(\hat{k})}{F_B(\hat{k})}[\hat{P}^{pu}(\hat{k}) - \psi^*(\hat{k})]$, so $(\psi^*)'(\hat{k}^-) = (\psi^*)'(\hat{k}^+)$, and ψ^* is indeed C^1 at $\hat{k}$.

Next, I verify that ψ^* satisfies (OPT-J) and (OPT-S). Starting with (OPT-S), for all k, $\psi^*(k) \geq P^{pu}(\underline{v})$. Hence, for any $w \geq 0$, $w(\psi^*(k) - P^{pu}(\underline{v})) \leq 0$. Likewise, for any $u \geq 0$, and $k \neq k^{pu}$,

$$\begin{aligned} u\left[\frac{f_B(k)}{F_B(k)}(P^{pu}(k) - \psi^*(k)) - (\psi^*)'(k)\right] - (r + \Lambda(k))\psi^*(k) \\ + \Lambda(k)\Pi_S(k) \end{aligned} \begin{cases} \leq 0, & k \in (\hat{k}, 1] \\ = 0, & k \in (k^{pu}, \hat{k}] \\ < 0, & k \in [\underline{v}, k^{pu}) \end{cases}$$

The weak inequality in the first case follows from Lemma 5 and minor algebra; the equality in the second case is by construction, since $P^{pu} = \hat{P}^{pu}$ for $k \in [k^{pu}, \hat{k}]$; the strict inequality follows from $\psi^*(k) = P^{pu}(k) = P^{pu}(\underline{v}) > R^{pu}(k)$ for $k < k^{pu}$. Hence, ψ^* satisfies (OPT-S).

To check (OPT-J), define

$$U^*(y; x) = (1 - F_B(y; x))P^{pu}(y) + F_B(y; x)\psi^*(y), \ \underline{v} \leq y \leq x \leq 1$$

For $y \in (k^{pu}, \hat{k})$, using $\psi^*(y) = R^{pu}(y)$, $P^{pu}(y) = \hat{P}^{pu}(y)$, and it follows that $\frac{\partial}{\partial k'} U_y^*(y; k) = (1 - F_B(y; k))\,(\hat{P}^{pu})'(y)$. Therefore, since $\hat{P}^{pu}$ is strictly increasing on $[k^{pu}, \hat{k}]$, and $U^*(y, x) = P^{pu}(\underline{v})$ for $y \leq k^{pu}$,

$$\max_{\underline{v} \leq y \leq \hat{k}} U^*(y; k) = L(k, \hat{k}). \tag{47}$$

In particular, (OPT-J) holds for $k \leq \hat{k}$.

Using the definition of ψ^* on $(\hat{k}, 1]$, for k', k such that $\hat{k} < k' < k$,

$$\begin{aligned} U^*(k'; k) &= (1 - F_B(k'; k))P^{pu}(k') \\ &\quad + F_B(k'; k)\left[(1 - F_B(\hat{k}; k'))\hat{P}^{pu}(\hat{k}) + F_B(\hat{k}; k')R^{pu}(\hat{k})\right] \\ &= (1 - F_B(\hat{k}; k))\hat{P}^{pu}(\hat{k}) + F_B(\hat{k}; k)R^{pu}(\hat{k}) = U^*(\hat{k}; k) \end{aligned} \tag{48}$$

where the last line uses $F_B(k';k)F_B(\hat{k};k') = F_B(k')F_B(\hat{k})/F_B(k)F_B(k')$ and $P^{pu}(k') = \hat{P}^{pu}(\hat{k})$ for $k' > \hat{k}$. (48) and the previous arguments yield $\max_{\underline{v} \leq y \leq k} U^*(y;k) = \psi^*(k)$ for all k, so (OPT-J) holds. By Proposition 2, the seller policy is a best response to P^{pu}.

A Online Appendix

A.1 Proof of Theorem 2

First, I define RME's of the hybrid starting-public game.

Definition 13. Consider a triple[33]

$$\Sigma = \left(\{\mathbf{K}^{k_0,k,t}\}_{k_0\in[0,1],k\leq k_0,t\geq 0^-}, \{\lambda_t^*[\cdot]\}_{t\geq 0}, P^{s.p.}(\cdot,\cdot)[\cdot]\right),$$

together with a value function $J_S^{s.p.}(\cdot,\cdot)[\cdot]$.

Let $\mathbf{P}_{t,k_0}(\mathbf{Q}) = \{P^{s.p.}(Q_s, t+s)[k_0]\}_{s\geq 0^-}$ and $\mathbf{L}^*_{t,k_0} = \{\lambda^*_{t+s}[k_0]\}_{s\geq 0^-}$ denote future the prices and costly intervention rates that Σ prescribes for time t onwards in response to a cutoff path $\mathbf{Q}$, given an initial jump to k_0 in the public phase. Then Σ is a Markov equilibrium of the starting-public game if

1. **Buyer Optimality** For all $k_0 \in [0,1]$, $k \leq k_0$, $t \geq 0^-$, accepting at accepting at $\tau^* = \inf\{s : P^{s.p.}(K^{k_0,k,t}_{t+s}, t+s)[k_0] \leq P^{s.p.}(v_B, t+s)[k_0]\}$ solves v_B's continuation problem
$$\sup_{\tau\geq 0} \mathcal{U}_B^{v_B}(\tau; \mathbf{P}_{t,k_0}(\mathbf{K}^{k_0,k,t}), \mathbf{L}^*_{t,k_0}).$$

2. **Seller Optimality**

   a. Start of Game For an admissible cutoff path $\mathbf{Q}$, let $\mathbf{Q}^+ = \{Q_t\}_{t\geq 0}$, i.e., $\mathbf{Q}^+ = \mathbf{Q}$ if $\mathbf{Q}$ does not jump at $t = 0$; otherwise $\mathbf{Q}^+$ is the continuation of $\mathbf{Q}$ after the initial jump. Then $\mathbf{K}^{k_0=1,k=1,t=0^-}$ solves the seller's initial problem
$$\sup_{\mathbf{Q}\in\mathcal{A}_1} (1 - F_B(Q_0))P^{s.p}(Q_0, 0)[Q_0] + F_B(Q_0)\mathcal{U}_S^{Q_0}(\mathbf{Q}, \mathbf{P}_{0,1}(\mathbf{Q}^+), \mathbf{L}^*_{0,1}),$$
   and delivers value $J^{s.p}(1,0)[1]$.

   b. Private Phase For any $k_0 \in [0,1]$, $k \in [0,k_0]$, and $t \geq 0$, $\mathbf{K}^{k_0,k,t}$ solves the seller continuation problem
$$\sup_{\mathbf{Q}\in\mathcal{A}_k} F_B(k)^{-1}\mathcal{U}_S^k(\mathbf{Q}, \mathbf{P}_{t,k_0}(\mathbf{Q}), \mathbf{L}^*_{t,k_0})$$
   and delivers value $J_S^{s.p}(k,t)[k_0]$.

5. **Entrant Optimality, Correct On-Path Private Phase Beliefs**
   For every $s \geq 0$, k_0, $\lambda^*_s[k_0]$ satisfies
$$\lambda^*_s[k_0] = \Lambda(K_s^{k_0,k=k_0,t=0^-}) = \ell_0 G(\Pi_A(K_s^{k_0,k=k_0,t=0^-})), \tag{49}$$
   i.e., the entrants have correct conjectures about the cutoffs that are supposed to follow every initial offer.

Proof of Theorem 2. Let $k^\diamond$ be the initial jump of some public offers RME. Since that jump ends on the smooth trade locus, the seller's value at $k = 1$ under public offers is

[33] To avoid duplicate notation, I use the symbol $\mathbf{K}^{k,t,k_0}$ evaluated at different superscripts to denote both the cutoff path at the start of the game (corresponding to $t = 0^-$) and all cutoff paths from the private phase. Note that the "$t = 0$" path is distinct from the $t = 0^-$ path: the former refers to the path at the very start of the private offers phase, at which point the seller no longer directly controls costly intervention rates, while the latter refers to the path at the start of the game, at which point the seller can control the initial costly intervention rate. Naturally, for $t = 0^-$, $k = k_0 = 1$, while for $t = 0$, $k = k_0$ (but $k_0 < 1$ is allowed).

$$U(1,k^\diamond) = (1 - F_B(k^\diamond))\,\hat{P}^{pu}(k^\diamond) + F_B(k^\diamond) D(k^\diamond)\Pi_S(k^\diamond).$$

I derive a lower bound on the seller's payoff in any RME of the starting-public game and prove that this bound exceeds $U(1,k^\diamond)$. A necessary condition for the initial public offer of that starting-public game is

$$K_0 \in \arg\max_{k_0\in[0,1]} \left\{ (1-F_B(k_0))\, P^{s.p.}(k_0, t=0)[k_0] + F_B(k_0) D^X(t=0|\boldsymbol{\lambda}^*[k_0])\Pi_S(k_0) \right\}.$$

Indeed, taking as given $P^{s.p.}(\cdot,\cdot)[\cdot]$ and $\boldsymbol{\lambda}^*[\cdot]$, the seller could always choose an impulse control for K_t with a different size of initial jump, and after the initial jump the seller faces a continuation game that looks like a private game with initial state $K_{0^-} = k_0$. In particular, this means that the seller's payoff in the starting-public game is at least

$$(1 - F_B(k^\diamond)) P^{s.p.}(k^\diamond, t=0)[k^\diamond] + F_B(k^\diamond) D^X(t=0|\boldsymbol{\lambda}^*[k^\diamond])\Pi_S(k^\diamond)$$

Consistency of entrant beliefs requires that, for $k_0 = k^\diamond$, $\lambda_0^*[k^\diamond] = \Lambda(k^\diamond)$. By Theorem 4, $D^X(t = 0|\boldsymbol{\lambda}^*[k^\diamond]) > D(k^\diamond)$. Adapting the arguments in Theorem 3 to a private game with initial state $K_{0^-} = k^\diamond$,

$$P^{s.p.}(k^\diamond, t=0)[k^\diamond] = D^X(t=0|\boldsymbol{\lambda}^*[k^\diamond])\pi_S(k^\diamond) > \hat{P}^{pu}(k^\diamond).$$

Altogether, comparing term by term,

$$(1 - F_B(k^\diamond))\, P^{s.p.}(k^\diamond, t=0)[k^\diamond] + F_B(k^\diamond) D^X(t=0|\boldsymbol{\lambda}^*[k^\diamond])\Pi_S(k^\diamond) > U(1,k^\diamond),$$

which proves the result. □

A.2 Construction and for Private Gap Case

The "right" construction, in terms of incentives that would support private on-path play similar to the public gap case—smooth trade, followed by an impasse of stochastic length, after which the pooling offer is made—is clear. The possibility of making a profitable pooling offer creates a region in (t, k) space inside which it is overwhelmingly tempting to make a pooling offer. The boundary of this region usually *cannot be crossed on path*, since prices would drop discontinuously and deterministically. Hence, in equilibrium th (e cutoff path must "reflect" off the boundary, which is achieved by the seller mixing between a "normal" offer and a discontinuously generous pooling offer, in such a way that the marginal type remains indifferent between accepting and rejecting and the cutoff path skirts that pooling region. Eventually, normal trade "stops," with the seller mixing between holding firm and making the pooling offer.

A full existence proof, however, raises some technical issues close to the frontier of the literature on control. The resulting value function for the seller is kinked in a way that makes standard viscosity approaches to verification (to my knowledge) inapplicable.[34]

[34] The notion of viscosity solution can be extended Hamiltonians with discontinuous data by considering the upper and lower continuous envelope of the Hamiltonion and demanding that the value function be a viscosity sub/supersolution of the respective envelope. That procedure can also be followed here to check that the value is a viscosity solution of the HJB in an appropriate sense, but what remains is to prove *uniqueness* of viscosity solutions to the HJB for such a setup—that would "certify" the optimality of the seller's policy for all states.

For an example of viscosity solutions with data discontinuous in the state, and the issues this raises, see Akian et al. (2001). To my knowledge, the control literature has tackled only these problems in parametric examples relevant to mathematical finance, and under some special assumptions on the nature of the discontinuity (Giga et al., 2011).

With that caveat in mind, I describe such a construction for the DEI case and the outstanding issues. To describe the results concisely, I require some auxiliary definitions. Let $k^{pr} = \sup\{k \in [\underline{v}, 1] : R^{pu}(k) = \underline{v} - D(k)\pi_B(\underline{v})\}$. By Assumption 3, $k^{pr} \in (\underline{v}, 1)$.[35] Let z^* be the unique value for which the unique forward solution to the dynamical system (33) starting at $(k_0 = 1, z_0 = z^*)$ exits the domain

$$\mathcal{D} = \{(k, z) : k \in (0, 1], D(0) < z < D(1)\},$$

through $(k^{pr}, D(k^{pr})$.[36] Let $K^\dagger$ and $Z^\dagger$ denote that solution, and let t^{pr} denote its (finite) time of exit from $\mathcal{D}$. Define the rate path λ_t^* as follows.

$$\lambda_t^* = \begin{cases} \frac{rZ_t^\dagger}{1-Z_t^\dagger}, t < t^{pr} \\ \Lambda(k^{pr}), t \geq t^{pr} \end{cases} \tag{50}$$

In addition, let the locii $\hat{k}(t)$ and $\tilde{k}(t)$ solve, respectively,

$$\begin{aligned} \frac{\lambda_t^*}{\lambda_t^* + r}\Pi_S(\hat{k}(t)) &= \underline{v} - D^X(t|\boldsymbol{\lambda}^*)\pi_B(\underline{v}) \\ D^X(t|\boldsymbol{\lambda}^*)\Pi_S(\tilde{k}(t)) &= \underline{v} - D^X(t|\boldsymbol{\lambda}^*)\pi_B(\underline{v}) \end{aligned} \tag{51}$$

and let $\hat{t}(k) := (\hat{k})^{-1}(t)$ for $t \leq t^{pr}$ denote the inverse of $\hat{k}(t)$. Note that the locii $\hat{k}, \tilde{k}$ are constant for $t \geq t^{pr}$, and are strictly decreasing (increasing) in (t, k) space for $t < t^{pr}$ if DEI (DDI) holds.

Below I use the notation

$$U^{pr}(k'; k, t)(1 - F_B(k'; k))P^{pr}(k, t) + F_B(k'; k)J^{pr}(k', t), \; k' \leq k$$

to denote the seller's payoffs from jumping the state from k to $k' \leq k$, given continuation payoffs of $J^{pr}(k, t)$. I will use repeatedly use the following fact:

> **<u>Fact $\star$</u>** If P^{pr} and J^{pr} are partially differentiable in k at (k', t), and $P^{pr}(k', t) = J^{pr}(k', t) + \frac{F_B(k')}{f_B(k')}J_k^{pr}(k', t)$, then $U_{k'}^{pr}(k'; k, t) = (1 - F_B(k'; k))P_k^{pr}(k', t)$.

Theorem 7 (DEI Construction)**.** *Consider the 4-tuple of $\Big((K^{k,t})_{k\in[0,1],t\geq 0^-}, P^{pr}(k,t), (\lambda_t^*)_{t\geq 0}, H\Big)$, where costly intervention rates are given by* (50)*, the law of motion for cutoffs and pooling offers at state (k, t) is given by*

1. *If $k < \tilde{k}(t)$, immediately make the pooling offer.*
2. *If $k > \tilde{k}(t)$, make no pooling offers, and trade smoothly according to*

$$\dot{k} = -\left[\frac{(\lambda_t^* + r)(k - P^{pr}(k, t)) + \frac{\partial}{\partial t}P^{pr}(k, t) - \lambda_t^*\pi_B(k)}{\frac{\partial}{\partial k}P^{pr}(k, t)}\right].$$

3. *If $t \geq t^{pr}$, $k = \tilde{k}(t)$, hold the cutoff constant forever while making the pooling offer at a Poisson rate γ given by*

$$k^{pr} - P^{pr}(k^{pr}, t^{pr}) = \frac{\lambda_{t^{pr}}^*\pi_B(k^{pr})}{\gamma + r + \lambda_{t^{pr}}^*} + \frac{\gamma(k^{pr} - P^{pr}(\underline{v}, t^{pr}))}{\gamma + r + \lambda_{t^{pr}}^*} \tag{52}$$

[35] In the DDI case, the equation $D(k)\Pi_S(k) = \underline{v} - D(k)\pi_B(\underline{v})$ has a unique solution, so the supremum is redundant. There may be multiple solutions in the DEI case, so it is possible that $k^{pr} \neq k^{pu}$ where the latter is as in Proposition 1. Even so, the payoff comparisons from the no-gap case will continue to hold, since they only apply for states (k, t) on the private offers path of play.

[36] Existence and uniqueness of such z^* and the associated forward solution to (33) follow from Lemma 10.

4. *If $k = \tilde{k}(t)$, $t < t^{pr}$, trade smoothly at the rate $\dot{K}_t = \tilde{k}'(t)$, and make the pooling offer at a Poisson rate γ_t satisfying*

$$(r + \lambda_t^* + \gamma_t)(k - P^{pr}(k,t)) = \lambda_t^* \pi_B(k) + \gamma_t(k - P^{pr}(\underline{v},t)) \\ - P_k^{pr}(k,t)\tilde{k}'(t) - P_t^{pr}(k,t). \quad (53)$$

and prices P^{pr} are given by

$$P^{pr}(\underline{v},t) := \underline{v} - D^X(t|\boldsymbol{\lambda}^*)\pi_B(\underline{v}). \quad (54)$$

$$P^{pr}(k,t) = \begin{cases} P^{pr}(\underline{v},t), & k < \tilde{k}(t). \\ D^X(t|\boldsymbol{\lambda}^*)\pi_S(k), & k \geq \tilde{k}(t). \end{cases} \quad (55)$$

These strategies are

- *Audience-optimal.*
- *Buyer-optimal.*
- *Satisfy the consistent-beliefs condition.*
- *Are sequentially optimal starting for the seller at $(k,t) \notin [k^{pr}, 1] \times [0, t^{pr}]$.*
- *Induce a seller value function J^{pr} that satisfies a rate-impulse-control quasi-variational inequality outside the $\tilde{k}(t)$ locus. Let $\mathcal{M}$ and $\mathcal{R}$ denote the operators*

$$\mathcal{R}^{u,\gamma}\varphi(k,t) = [P^{pr}(k,t) - \varphi(k,t)]\frac{f_B(k)}{F_B(k)}u + \frac{\partial}{\partial k}\varphi(k,t)(-u) \\ + \lambda_t^*[\Pi_S(k) - \varphi(k,t)] + \gamma[P^{pr}(\underline{v},t) - \varphi(k,t)] \\ + \frac{\partial}{\partial t}\varphi(k,t) - r\varphi(k,t)$$

and

$$\mathcal{M}\varphi(k,t) = \max_{k' \leq k}\{ \quad (1 - F_B(k';k))P^{pr}(k,t) + F_B(k';k)\varphi(k',t)\} \quad (56)$$

that act on C^1 functions $\varphi : [0,1] \times \mathbb{R}_+ \to \mathbb{R}$. Then J^{pr} satisfies

$$0 = \max\left\{ \sup_{|\dot{k}| \geq 0, \gamma \geq 0} \left\{\mathcal{R}^{|\dot{k}|,\gamma} J^{pr}(k,t) - rJ^{pr}(k,t)\right\}, \quad \mathcal{M}J^{pr}(k,t) - J^{pr}(k,t)\right\} \quad (57)$$

in a classical sense for all (k,t) such that $k \neq \tilde{k}(t)$.

- *For (k,t) such that $k = \tilde{k}(t)$, J^{pr} is continuous but not C^1.*

The construction has the "right" incentives—with the seller preserving the buyer's incentive constraints mixing between a regular offer and a pooling offer/reflecting off a boundary—but falls partly short of a full equilibrium proof.

The behavior in the Theorem (i) induces a value function function for the seller that is not C^1, (ii) the discontinuities in the partial derivatives are on a locus that the seller can cross from above and from below, (iii) that non-differentiable locus is not on vertical or horizontal lines in (t,k) space, but is strictly decreasing, and (iv) the cutoff path prescribed for the seller spends positive time on that non-C^1 locus. Because of point (i) the vanilla verification theorem for seller optimality that I used in the no-gap case will not work. Because of point (ii), the approach I take in the gap public game, splitting the state space into regions that are accessed in a fixed order, also will not work. Because of points (iii) and (iv), the problem requires a viscosity approach, but even then "standard" viscosity approaches do not readily apply since the data in the HJB is *discontinuous in the state* (since the price $P^{pr}(k,t)$ is) and the HJB contains a non-local operator.

Proof. I introduce some notation for the different regions of the state space:

- $A_1 = \{(k,t) : k > \tilde{k}(t), t < t^{pr}\}$.
- $A_2 = \{(k,t) : k > \tilde{k}(t), t \geq t^{pr}\}$.
- $B = \{(k,t) : k \in (k^{pr}, \tilde{k}(t)), t \leq t^{pr}\}$
- $C = \{(k,t) : k \leq k^{pr}\}$.

Audience optimality and consistent beliefs follow by construction of (50). To check buyer optimality for $(k,t) \in \text{int}\{B \cup C\}$, notice that $v = \underline{v}$ is by construction indifferent between accepting and rejecting $P^{pr}(\underline{v},t)$, at each t. Given the expectation of an immediate offer of $P^{pr}(\underline{v},t)$, all types will therefore have weak incentives to accept immediate. The Poisson rate γ in (52) makes k^{pr} indifferent between accepting $P^{pr}(k^{pr})$ and waiting for the pooling offer, while, by the same arguments made in no-gap smooth trading, the Poisson rates γ_t and speed of trade $\tilde{k}'(t)$ in (53) make $k = \tilde{k}(t)$ just indifferent between accepting and rejecting. Finally, the marginal type is, by construction, indifferent between accepting and not at all points in $A_1 \cup A_2$. (The smooth trade speeds can be shown to be positive by Assumption 3). Altogether, by the skimming property, and using the buyer optimality arguments from the no-gap case with minimal modification, buyer optimality holds here, too.

Seller optimality within C One has $J^{pr}(k,t) = P^{pr}(\bar{v},t)$, so $J^{pr}(k,t) = \mathcal{M}J^{pr}(k,t)$ follows trivially. Plugging in $J^{pr}(k,t) = P^{pr}(\underline{v},t) = P^{pr}(k,t) = \underline{v} - D^X(t|\boldsymbol{\lambda}^*)\pi_B(\underline{v})$, one has, by simple calculus,

$$
\begin{aligned}
\mathcal{R}^{u,\gamma}J^{pr}(k,t) &= \lambda_t^*[\Pi_S(k) - P^{pr}(\underline{v},t)] + \frac{\partial}{\partial t}P^{pr}(\underline{v},t) - rP^{pr}(\underline{v},t) && (58)\\
&= (\lambda_t^* + r)\left[\frac{\lambda_t^*}{\lambda_t^* + r}(\Pi_S(k) + \pi_B(\underline{v})) - \bar{v}\right] && (59)\\
&\leq (\lambda_t^* + r)\left[D^X(t|\boldsymbol{\lambda}^*)(\Pi_S(k) + \pi_B(\underline{v})) - \underline{v}\right] \leq 0
\end{aligned}
$$

where the inequality follows from $k \leq k^{pr} \leq \tilde{k}(t)$. Hence $J^{pr}(k,t)$ solves (65) classically for $C \setminus \{(k,t) : k \in [k^{pr}\}$. The seller's policy is therefore optimal in C by verification arguments that are nearly identical to those in Proposition 2, and are therefore omitted.

Seller optimality within B By identical arguments to the those for region C, $J^{pr}(k,t) = P^{pr}(\underline{v},t)$ solves (65) classically in B.

Seller optimality within A_2 The seller's payoffs are $J^{pr}(k,t) = D^X(t|\boldsymbol{\lambda}^*)\Pi_S(k)$, using the same arguments as before. Since $k > \tilde{k}(t)$, $J^{pr}(k,t) > P^{pr}(\underline{v},t)$. This, together with $J^{pr}(k',t) = P^{pr}(\underline{v},t)$ for $t \geq t^{pr}, k' \leq k^{pr}$, yields $J^{pr}(k,t) \geq U^{pr}(k';k,t)$ for $(k,t) \in A_2,$, $k' \leq k^{pr}$. Moreover, $P^{pr}(k,t)$ is strictly increasing in k within A_2 and satisfies $P^{pr}(k,t) = J^{pr}(k,t) + \frac{F_B(k)}{f_B(k)}J_t^{pr}(k,t)$, so using the key fact at the beginning of the proof, $J^{pr}(k,t) = \sup_{k' \in (k^{pr},k]} U^{pr}(k';k,t)$ $(J^{pr}(k,t)$. Altogether, $J^{pr}(k,t) = \mathcal{M}J^{pr}(k,t)$; this fact, together with $P^{pr}(k,t) = J^{pr}(k,t) + \frac{F_B(k)}{f_B(k)}J_t^{pr}(k,t)$, implies that $J^{pr}(k,t)$ solves (65) classically in A_2.

Finally, starting from A_2, the seller can only enter C. Given the optimality results for region C, verification arguments from Proposition 2 (with minor modifications that again omitted) once again imply that the seller's policy is optimal in $A_2 \cup C$.

Seller optimality within A_1 By identical arguments to the those for region A_2, $J^{pr}(k,t) = D^X(t|\boldsymbol{\lambda}^*)$ and $J^{pr}(k,t) \geq P^{pr}(\underline{v},t) \geq U^{pr}(k';k,t)$ $k' \leq \tilde{k}(t)$. Since $P^{pr}(k',t)$ is strictly increasing in k' for $(k',t) \in A_1$, Fact $\star$ above implies that $J^{pr}(k,t) = \max_{k' \in (\tilde{k}(t),k]} U^{pr}(k',k,t)$. Altogether, $J^{pr}(k,t) = \mathcal{M}J^{pr}(k,t)$, which given the smooth-trade expression for J^{pr} implies that J^{pr} satisfies (65) classically in A_1.

Off-Path Stopped Clock Offers This construction is standard as in discrete time (see Chaves and Varas (2022) for a continuous time version). At points of discontinuity of P^{pr}, by construction the seller is indifferent between the endpoints. Hence it suffices to consider a two point lottery between the high and the low offer, with probabilities that make the marginal type indifferent. $\square$

A.3 DDI Case (Sketch)

The DDI has the additional wrinkle that the boundary of the state space at which the pooling offer becomes tempting for the seller will typically be *increasing*. This means that, even if the seller refuses to trade, that region will still be entered at a positive speed. To preserve incentives for the buyer, prices in those regions that must be entered at positive speed must move continuously, i.e., must match the pooling offer at that boundary. Following that train of thinking, the natural equilibrium to construct would be one in which there is smooth trade with an eventual impasse on path (as in the DEI case), and the problematic part of the pooling region is approached smoothly off path. This seems to require an enlargement of the seller's strategy space: there are off path regions where, if one follows the construction strategy above, reservation prices are discontinous, and the seller must mix *between prices* while holding the *cutoff* constant. Hence, off path, behavior is not Markov in (k,t) but must also depend on the lowest price charged. For completeness, I provide a heuristic description of the off path play that still captures the right incentives.

Heuristically, we can think of the seller as being able to drive jumps in the price using a controlled Markov process, as in Online Appendix A.4.

The buyer again chooses a reservation price strategy, now given by a function $\kappa(p,t)$ that gives the lowest type to accept a price p at time t. the seller's objective is

$$\sup_{P \in \mathcal{A}_p^{DDI}} \mathbb{E}_t \left[\int_t^\sigma e^{-r(s-t)} P_s d(1 - F_B(\kappa(P_s, t+s); k)) \right. \\ \left. + e^{-r(\sigma-t)} F_B(\kappa(P_{\sigma-t},\sigma);k) \Pi_S(\kappa(P_{\sigma-t},\sigma)) \right] \quad (60)$$

For states where P^{pr} is strictly continuous in k, the game can be analyzed in quantity space just as before. The new formulation allows the seller to stochastically jump the price without jumping the cutoff at points where P^{pr} is discontinuous, which cannot be formalized in quantity space and will be necessary to preserve incentives off path.

Consider the price path P, reservation strategy κ and costly intervention rates λ_t^* such that

- rates are given by (50).
- $\kappa(p,t) = (P^{pr})^{-1;k}(p,t)$ where $(P^{pr})^{-1;k}$ is the generalized inverse with respect to the first argument, and

$$P^{pr}(\underline{v}, t) = \underline{v} - D^X(t|\boldsymbol{\lambda^*})\pi_B(\underline{v}). \tag{61}$$

$$P^{pr}(k,t) = \begin{cases} P^{pr}(\underline{v}, t), & k < \hat{k}(t). \\ D^X(t|\boldsymbol{\lambda^*})\pi_S(k), & k > k^{pr}. \\ \displaystyle\int_t^{\hat{t}(k)} \lambda_s^* e^{-\int_t^s (r+\lambda_\nu^*)d\nu} ds \pi_S(k) & \\ + e^{-\int_t^{\hat{t}(k)}(r+\lambda_s^*)ds} P^{pr}(\underline{v}, \hat{t}(k)), & (k,t) \in [\hat{k}(t), k^{pr}) \times [0, t^{pr}] \end{cases} \tag{62}$$

- For $(p,t) \notin \{(p',t') : P^{pr}(k^{pr}+, t') = p', t' < t^{pr}\}$, $P_t = P^{pr}(K_t, t)$, where K_t moves according to

  1. If $K_t \le \hat{k}(t)$, immediately make the pooling offer, i.e. drop k to $\underline{v}$ and charge $P^{pr}(\underline{v}, t)$.
  2. If $K_t > k^{pr}$ or $K_t \in (\hat{k}(t), k^{pr}), t < t^{pr}$, make no pooling offers, and trade smoothly according to

  $$\dot{K}_t = -\left[\frac{(\lambda_t^* + r)(K_t - P^{pr}(K_t,t)) + \frac{\partial}{\partial t}P^{pr}(K_t,t) - \lambda_t^*\pi_B(K_t)}{\frac{\partial}{\partial k}P^{pr}(K_t,t)}\right].$$

  3. If $t \ge t^{pr}$, $K_t = k^{pr}$, hold the cutoff constant forever while making the pooling offer $P^{pr}(\underline{v}, t)$ at a Poisson rate γ given by

  $$P^{pr}(k^{pr}, t^{pr}) = \left(1 - \frac{\gamma}{\gamma + r}\right) D(k^{pr})\pi_B(k^{pr}) + \frac{\gamma}{\gamma + r} P^{pr}(\underline{v}, t^{pr})$$

- If $p = P^{pr}(k^{pr}+, t)$ for $t < t^{pr}$, keep the deterministic part of the price path constant until t^{pr}, while jumping the price down to $P^{pr}(k^{pr}-, t)$ at a Poisson rate η_t given by

$$\begin{aligned}(r + \lambda_t^* + \eta_t)(k^{pr} - P^{pr}(k^{pr}+, t)) = \lambda_t^*\pi_B(k^{pr}) \\ + \eta_t(k^{pr} - P^{pr}(k^{pr}-, t)) - P_t^{pr}(k^{pr}+, t)\end{aligned} \tag{63}$$

These strategies are

- Audience-optimal.
- Buyer-optimal.
- Satisfy the consistent-beliefs condition.
- Are sequentially optimal for the seller starting at states (k,t) with $k < \hat{k}(t)$
- Outside of $k = k^{pr}, t < t^{pr}$ and $(k,t) = (\hat{k}(t), t)$ the strategies induce an equivalent quantity space seller value function J^{pr} that satisfies a rate-impulse-control quasi-variational inequality outside the $\tilde{k}(t)$ and $\hat{k}(t)$. Let $\mathcal{M}$ and $\mathcal{R}$ denote the operators

  $$\mathcal{M}\varphi(k,t) = \max_{k' \le k}\{ (1 - F_B(k';k))P^{pr}(k,t) + F_B(k';k)\varphi(k',t)\} \tag{64}$$

  and

  $$\begin{aligned}\mathcal{R}^{u,\gamma}\varphi(k,t) = [P^{pr}(k,t) - \varphi(k,t)]\frac{f_B(k)}{F_B(k)}u + \frac{\partial}{\partial k}\varphi(k,t)(-u) \\ + \lambda_t^*[\Pi_S(k) - \varphi(k,t)] + \gamma[\mathcal{M}\varphi(k,t) - \varphi(k,t)] \\ + \frac{\partial}{\partial t}\varphi(k,t) - r\varphi(k,t)\end{aligned}$$

  that act on C^1 functions $\varphi : [0,1] \times \mathbb{R}_+ \to \mathbb{R}$. Then J^{pr} satisfies

$$0 \quad = \quad \max\left\{ \begin{array}{r} \sup_{|\dot{k}|\geq 0,\gamma\geq 0} \left\{ \mathcal{R}^{|\dot{k}|,\gamma} J^{pr}(k,t) - rJ^{pr}(k,t) \right\}, \\ \mathcal{M}J^{pr}(k,t) - J^{pr}(k,t) \end{array} \right\} \tag{65}$$

- For (k,t) such that $k \in \{k^{pr}, \hat{k}(t)\}$, J^{pr} is continuous but not C^1.

Proof. I introduce some notation for the different regions of the (k,t) state space:

- $A_1 = \{(k,t) : k > k^{pr}, t \geq t^{pr}\}$.
- $A_2 = \{(k,t) : k > k^{pr}, t < t^{pr}\}$.
- $B = \{(k,t) : k = k^{pr}, t \leq t^{pr}\}$.
- $C = \{(k,t) : k \in (\hat{k}(t), k^{pr}), t \leq t^{pr}\}$
- $D = \{(k,t) : k \leq \hat{k}(t)\}$.

For all regions outside B, verification can proceed in quantity space, similar to the DEI case.

Audience optimality and consistent beliefs follow by construction of (50). Buyer optimality in D follows as before. By the usual smooth-screening arguments, the speed of trade $\dot{k}$ and prices P^{pr} in C, given buyer optimality in D, makes the marginal type indifferent within C. For region B, given continuation play within $C \cup D$, the mixing rate (63) makes $v_B = k^{pr}$ just indifferent between accepting $P^{pr}(k^{pr}+, t)$ and waiting for $P^{pr}(k^{pr}-, t)$ at state (k^{pr}, t): integrating equation (63) backwards from t^{pr},

$$\begin{aligned} k^{pr} - P^{pr}(k^{pr}+, t) = &\int_t^{t^{pr}} \lambda_s^* e^{-\int_t^s (r++\eta_\nu+\lambda_\nu^*)d\nu} ds \pi_B(k^{pr}) \\ &+ \int_t^{t^{pr}} \eta_s e^{-\int_t^s (r++\eta_\nu+\lambda_\nu^*)d\nu} \left(k^{pr} - P^{pr}(k^{pr}-, s)\right) ds \\ &\qquad + e^{-\int_t^{t^{pr}} (r++\eta_s+\lambda_s^*)ds} (k^{pr} - P^{pr}(k^{pr}, t^{pr})), \end{aligned}$$

and $P^{pr}(k^{pr}+, t)$ is exactly k^{pr}'s reservation price at (k^{pr}, t) Buyer optimality then follows by identical arguments to the DEI case and the skimming property (the smooth trade speeds are positive using Assumption 3).

Seller optimality within D Plugging in $J^{pr}(k,t) = P^{pr}(\bar{v}, t) = \underline{v} - D^X(t|\boldsymbol{\lambda}^*)\pi_B(\underline{v})$, one has, by simple calculus,

$$\begin{aligned} \mathcal{R}^{u,\gamma} J^{pr}(k,t) &= \lambda_t^*[\Pi_S(k) - P^{pr}(\underline{v}, t)] + \frac{\partial}{\partial t} P^{pr}(\underline{v}, t) - rP^{pr}(\underline{v}, t) \\ &= \lambda_t^* \Pi_S(k) \lambda_t^* \pi_B(\underline{v}) - (\lambda_t^* + r)\pi_B(\underline{v}) \leq 0 \end{aligned} \tag{66}$$

where I have used $P^{pr}(k,t) = P^{pr}(\underline{v}, t)$ and $k \leq \hat{k}(t)$. Hence $J^{pr}(k,t)$ solves (65) classically in D. The remaining arguments are identical to the DEI case, and omitted.

Seller optimality within C Consider the function

$$N(k,t) = \int_t^{\hat{t}(k)} \lambda_s^* e^{-\int_t^s (r+\lambda_\nu^*)d\nu} ds \Pi_S(k) + \, e^{-\int_t^{\hat{t}(k)} (r+\lambda_s^*)ds} P^{pr}(\underline{v}, \hat{t}(k)). \tag{67}$$

These are the payoffs to the seller from stalling at (k,t) until reaching the locus $(k, \hat{t}(k))$. Partially differentiating yields

$$
\begin{aligned}
rN(k,t) &= \lambda_t^*(\Pi_S(k) - N(k,t)) + N_t(k,t) \\
P^{pr}(k,t) &= N(k,t) + \frac{F_B(k)}{f_B(k)} N_k(k,t)
\end{aligned}
\quad . \tag{68}
$$

where the second line uses

$$
\frac{\partial}{\partial \hat{t}(k)} N(k,t) = e^{-\int_t^{\hat{t}(k)}(r+\lambda_\nu^*)d\nu} ds \Big[\lambda_{\hat{t}(k)}^* \Pi_S(k) - (r + \lambda_{\hat{t}(k)}^*) P^{pr}(\underline{v}, \hat{t}(k)) \\
+ P_t^{pr}(\underline{v}, \hat{t}(k)) \Big] = 0 \tag{69}
$$

Therefore the seller is indifferent between trading at the required smooth speed and stalling until $t = \hat{t}(k)$. By the arguments in the no-gap case, the seller's payoffs at (k,t) are then given by $J^{pr}(k,t) = N(k,t)$. Using (69) one has that $N_k(k,t) > 0$, so

$$
N(k,t) > N(\hat{k}(t), t) = P^{pr}(\underline{v}, t). \tag{70}
$$

By direct differentiation, $P_k^{pr}(k,t) > 0$ in D, so using (68) and Fact $\star$, $U_{k'}^{pr}(k';k,t) > 0$ for all $k' \geq \hat{k}(t)$ and $J^{pr}(k,t) = \max_{k' \in (\hat{k}(t),k]} U^{pr}(k';k,t)$. For $k \leq \hat{k}(t)$, $U^{pr}(k';k,t) = P^{pr}(\underline{v},t)$, so using (70), it follows that $J^{pr}(k,t) = \mathcal{M}J^{pr}(k,t)$. Combining this last equality with (70) and (68), $R^{u,\gamma} J^{pr}(k,t) = 0$, and J^{pr} solves (65) classically inside this region.

The seller policy is sequentially optimal in D, and starting in C the cutoff can only enter D, spending zero time on the boundary. Once again verification arguments from Proposition 2, with minor modifications that are omitted, imply that the seller's policy is optimal in $C \cup D$.

Seller optimality for B The seller can only enter C or D from B. First, I show that the seller is willing to mix and enter C as required. By familiar arguments, given a price $P^{pr}(k^{pr}+,t) = D^X(t|\boldsymbol{\lambda}^*)\pi_S(k^{pr})$, the payoffs from a quiet period in B (leaving the price at $P^{pr}(k^{pr}+),t)$ with probability 1) equal $J^{pr}(k^{pr}+,t) = D^X(t|\boldsymbol{\lambda}^*)\Pi_S(k^{pr})$. Meanwhile, taking limits of the payoffs in C as $k \nearrow k^{pr}$, one obtains $J^{pr}(k^{pr}-,t) = D^X(t|\boldsymbol{\lambda}^*)\Pi_S(k^{pr}-) = J^{pr}(k^{pr}+,t)$. In particular, $\mathcal{M}J^{pr}(k^{pr}+,t) = J^{pr}(k+,t)$.

Seller optimality within A_2 This is identical to the DEI case, and thus omitted.

Seller optimality within A_1 By identical arguments to the those for region A_2 in the DEI case, $J^{pr}(k,t) = D^X(t|\boldsymbol{\lambda}^*)$ and $J^{pr}(k,t) \geq U^{pr}(k';k,t)$ for $k' \in [0,\hat{k}(t)] \cup \{k^{pr}\}$. Since $P^{pr}(k',t) =$ for $k' \in (\hat{k}(t), k^{pr})$, Fact $\star$ above implies that $\max_{k' \in (\hat{k}(t),k^{pr})} U^{pr}(k',k,t) = J^{pr}(k^{pr}-,t)$. Altogether, $J^{pr}(k,t) = \mathcal{M}J^{pr}(k,t)$, which given the smooth-trade expression for J^{pr}, implies that J^{pr} satisfies (65) classically in A_1.

Off-path stopped clock offers As in the DEI case, using two-point distributions. □

A.4 General $\hat{P}^{pu}$: Removing Assumption 4.1

I show how to extend the equilibrium construction to arbitrary $\hat{P}^{pu}$'s. I impose only the condition that $\hat{P}^{pu}$ has *finitely many local maxima*.

The construction requires richer jumping behavior, which can be formalized in a space larger than that in Definition 10.

Definition 14 (Strategy Space NP). To allow the seller to mix over cutoff paths, enlarge the probability space to include a controlled Poisson process X_t with intensity γ_t. Say a cutoff path $t \mapsto K_t$ is admissible if it is non-increasing and can be written as

$$K_t = K_{0^-} + \int_0^t \dot{K}_u du + \int_0^t \Delta K^s_{u^-} dX_u + \sum_{u \leq t} \Delta K^d_{u-}$$

where $(\dot{K}_t, \Delta K^s_t, \Delta K^d_t, \gamma_t)$ are $\mathcal{F}^X_t$-predictable processes, with $\mathcal{F}^X_t$ denoting the filtration generated by $(X_t)_{t\geq 0}$ $\mathcal{A}^{NP}_k$ denotes the admissible paths with initial value $K_{0^-} = k$.

In words, the seller can jump the cutoff both deterministically (represented by $\Delta^d_{u^-}$'s) and stochastically (represented by $\Delta^s_{u^-}$'s), with the stochastic jumps given by the ticks of the Poisson process controlled by the seller. The seller's payoff for a realization of a cutoff path $\mathbf{Q}$ is given by

$$\Pi(\mathbf{Q}) \qquad = \qquad \mathbb{E}^{\mathbf{Q}}_k \left[\int_0^\sigma e^{-rt} P^{pu}(Q_t) d(1 - F_B(Q_t; k)) \quad + \quad e^{-r\sigma} F_B(Q_\sigma; k) \Pi_S(Q_\sigma) \right]$$

where the expectation is over the time of costly intervention σ induced by $\mathbf{Q}$. and the seller's strategy solves $\sup_{\mathbf{Q}\in\mathcal{A}^{NP}_k} \mathbb{E}[\Pi(\mathbf{Q})]$. The buyer payoffs and full definition of a Weak Markov equilibrium are essentially as in the previous section, with minimal notational changes left to the reader.

To present the construction succinctly, I develop some necessary notation. Let $\eta(k) := \inf\{k' > k : L(k', k) = R(k')\}$ (with $\eta(1) := 1$ if $L(k', k) > R(k')$ for all $k' > k$). Let $N_\varepsilon(x)$ denote an $\varepsilon > 0$ ball around x. Then $\mathcal{S}(k) := \inf\{k' > k : \exists \varepsilon > 0 \text{ s.t. } \hat{P}^{pu}(k') \geq \hat{P}^{pu}(k) \forall k \in N_\varepsilon(k')\}$. In words, $\mathcal{S}(k)$ is the nearest local maximum of $\hat{P}^{pu}$ to the right of k.

The equilibrium reservation price curve, denoted $\breve{P}$, is constructed from $\hat{P}^{pu}$ and R^{pu} recursively as follows:

1. $\breve{P}(\underline{v}) = \hat{P}^{pu}(\underline{v})$.
2. If $\breve{P}(k) = \hat{P}^{pu}(k)$, $(\hat{P}^{pu})'(k) > 0$,
$$\breve{P}(k') = \hat{P}^{pu}(k') \forall k' \in [k, \mathcal{S}(k)].$$
3. If $\breve{P}(k) = \hat{P}^{pu}(k)$, $(\hat{P}^{pu})'(k) \leq 0$, $\eta(k) < 1$,
$$\breve{P}(k') = \begin{cases} \breve{P}(k), & \forall k' \in [k, \eta(k)) \\ \hat{P}^{pu}(\eta(k)), & k' = \eta(k), \eta(k) < 1 \end{cases}$$
4. If $\breve{P}(k) = \hat{P}^{pu}(k)$, $(\hat{P}^{pu})'(k) \leq 0$, $\eta(k) = 1$,
$$\breve{P}(k') = \breve{P}(k), \quad \forall k' \in [k, 1].$$

In particular, $\breve{P}$ can be discontinuous at $\eta(k)$ if $(\hat{P}^{pu})'(k) \leq 0$, and when $\breve{P}$ is strictly increasing, it must match $\hat{P}^{pu}$.

Proposition 3. *With the seller strategy space in Definition 14, there exists an equilibrium with the following on-path outcome:*

1. $P^{pu} = \breve{P}, \mathcal{L} = \Lambda$.
2. *At any k with $\breve{P}(k) > \breve{P}(k-)$, there is a quiet period with Poisson duration of rate $\gamma(k)$, where $\delta(k) := \gamma(k)/(\gamma(k) + r)$ solves*
$$k - \breve{P}(k) = (1 - \delta(k)) D(k) \pi_B(k) + \delta(k) \left(k - \breve{P}(k-)\right)$$

When the quiet period ends, the state jumps down to

$$\xi(k) := \inf\{k' < k : \breve{P}(k') = \breve{P}(k-)\}$$

The seller's payoff at such a k *is* $R^{pu}(k)$.

3. *If* $\breve{P}(k)$ *is constant immediately to the left of* k*, the state jumps down to* $\xi(k)$*, and the seller's payoff equals* $L(k, \xi(k))$.
4. *If* $\breve{P}$ *is strictly increasing and continuous at* k*, trade happens smoothly at a speed* $\max\{0, -\varphi^{pu}(k)\}$*, and the seller's payoff equals* $R^{pu}(k)$.

The verification for the seller follows almost identical steps to Proposition 2, and for the buyer it follows Theorem 5, so I omit them. Meanwhile, private offer payoffs equal $J^{priv}(k,t) = D^X(t|\boldsymbol{\lambda}^*)$ regardless of the shape of $\hat{P}^{pu}$, so the payoff characterization behind Theorems 1 and 2 hold in this very general environment.

Remark 5. If $\hat{P}^{pu}$ is in this general case, but there is a gap, then a similar construction of $\breve{P}$ holds, replacing step 1 in the construction with

- $\breve{P}(k) = \underline{v} - D(\underline{v})\pi_B(\underline{v}), \quad k \in [\underline{v}, k^{pu})$.
- $\breve{P}(k^{pu}) = \hat{P}^{pu}(k^{pu})$.